CYCLOTOMIC WENZL ALGEBRAS

SUSUMU ARIKI, ANDREW MATHAS, AND HEBING RUI

Abstract. Nazarov [Naz96] introduced an infinite dimensional algebra, which he called the *affine Wenzl algebra*, in his study of the Brauer algebras. In this paper we study certain "cyclotomic quotients" of these algebras. We construct the irreducible representations of these algebras in the generic case and use this to show that these algebras are free of rank $r^n(2n-1)!!$ (when Ω is $\mathbf{u}$–admissible). We next show that these algebras are cellular and give a labelling for the simple modules of the cyclotomic Wenzl algebras over an arbitrary field.

On the occasion of Professor George Lusztig's 60th birthday

1. Introduction

The Brauer algebras were introduced by Richard Brauer [Bra37] in his study of representations of the symplectic and orthogonal groups. In introducing these algebras Brauer was motivated by Schur's theory (see [Gre80]), which links the representation theory of the symmetric group $\mathfrak{S}_n$ and the general linear group $\mathrm{GL}(V)$ via their commuting actions on "tensor space" $V^{\otimes n}$, where $\mathfrak{S}_n$ acts by place permutations. The image of $\mathrm{GL}(V)$ in $\mathrm{End}(V^{\otimes n})$ is known as a *Schur algebra*. Analogously, the Brauer algebras are the images of symplectic or orthogonal groups in $\mathrm{End}(V^{\otimes n})$, where V is the defining representation for one of these groups.

The Brauer algebras have now been studied by many authors and they have applications ranging from Lie theory, to combinatorics and knot theory; see, for example, [BW89, Bro56, DWH99, Eny04, FG95, HW89a, HW89b, Jon94, Mar96, Naz96, Rui05, Ter01, Wen88]. In this paper we are interested not so much in the Brauer algebra itself but in affine and cyclotomic analogues of it. Our starting point is a (special case of) Nazarov's [Naz96] affine Wenzl algebra $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$.

Let R be a commutative ring. The representation theory of the affine Wenzl algebras $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$, where $\Omega = \{\,\omega_a \in R \mid a \geq 0\,\}$, has not yet been studied. Motivated by the theory of the affine Hecke algebras and the cyclotomic Hecke algebras of type $G(r,1,n)$ [Ari96, DJM99, Kle05] we introduce a "cyclotomic" quotient $\mathscr{W}_{r,n}(\mathbf{u}) = \mathscr{W}_n^{\mathrm{aff}}(\Omega)/\langle\prod_{i=1}^r(X_1-u_i)\rangle$ of $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$, which depends on an r–tuple of parameters $\mathbf{u} = (u_1,\dots,u_r) \in R^r$. We call $\mathscr{W}_{r,n}(\mathbf{u})$ a *cyclotomic Wenzl algebra*. This paper develops the representation theory of the algebras $\mathscr{W}_{r,n}(\mathbf{u})$.

The first question that we are faced with is whether the cyclotomic Wenzl algebra $\mathscr{W}_{r,n}(\mathbf{u})$ is always free as an R–module. The Brauer algebra $\mathscr{B}_n$ is free of rank $(2n-1)!! = (2n-1)\cdot(2n-3)\cdots 3\cdot 1$. We expect that the cyclotomic Wenzl algebra $\mathscr{W}_{r,n}(\mathbf{u})$ should be free of rank $r^n(2n-1)!!$. In section 3, a detailed study of the representation theory of $\mathscr{W}_{r,2}(\mathbf{u})$ shows that, in the semisimple case, $\mathscr{W}_{r,2}(\mathbf{u})$ has

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rank $r^2(2n-1)!!|_{n=2}$ if and only if Ω is $\mathbf{u}$–admissible. This constraint on Ω involves Schur's q–functions; see Definition 3.5. Our first main result is the following.

Theorem A. *Let R be a commutative ring in which 2 is invertible. Suppose that $\mathbf{u} \in R^r$ and that Ω is $\mathbf{u}$–admissible. Then the cyclotomic Wenzl algebra $\mathscr{W}_{r,n}(\mathbf{u})$ is free as an R–module of rank $r^n(2n-1)!!$.*

The proof of this result occupies a large part of this paper. The idea behind the proof comes from [AK94]: for "generic" R we explicitly construct a class of irreducible representations of $\mathscr{W}_{r,n}(\mathbf{u})$ and use them to show that the dimension of $\mathscr{W}_{r,n}(\mathbf{u})/\operatorname{Rad}\mathscr{W}_{r,n}(\mathbf{u})$ is at least $r^n(2n-1)!!$. It is reasonably easy to produce a set of $r^n(2n-1)!!$ elements which span $\mathscr{W}_{r,n}(\mathbf{u})$, so this is enough to prove Theorem A. We construct these irreducible representations by giving "seminormal forms" for them (Theorem 4.13); that is, we give explicit matrix representations for the actions of the generators of $\mathscr{W}_{r,n}(\mathbf{u})$. The main difficulty in this argument is in showing that these matrices respect the relations of $\mathscr{W}_{r,n}(\mathbf{u})$, we do this using generating functions introduced by Nazarov [Naz96].

The next main result of the paper shows that $\mathscr{W}_{r,n}(\mathbf{u})$ is a cellular algebra in the sense of Graham and Lehrer [GL96]. As a consequence we can, in principle, construct all of the irreducible representations of $\mathscr{W}_{r,n}(\mathbf{u})$ over an arbitrary field. Moreover, the decomposition matrix of $\mathscr{W}_{r,n}(\mathbf{u})$ is unitriangular.

Theorem B. *Suppose that 2 is invertible in R and that Ω is $\mathbf{u}$–admissible. Then the cyclotomic Wenzl algebra $\mathscr{W}_{r,n}(\mathbf{u})$ is a cellular algebra.*

We prove Theorem B by constructing a cellular basis for $\mathscr{W}_{r,n}(\mathbf{u})$. We recall the definition of a cellular basis in section 6; however, for the impatient experts we mention that the cell modules of $\mathscr{W}_{r,n}(\mathbf{u})$ are indexed by ordered pairs (f,λ), where $0 \le f \le \lfloor \frac{n}{2} \rfloor$ and λ is a multipartitions of $n-2f$, where $0 \le f \le \lfloor \frac{n}{2} \rfloor$, and the bases of the cell modules are indexed by certain ordered triples which are in bijection with the n–updown λ–tableaux.

Finally we consider the irreducible $\mathscr{W}_{r,n}(\mathbf{u})$–modules over a field R. The cell modules of $\mathscr{W}_{r,n}(\mathbf{u})$ have certain quotients $D^{(f,\lambda)}$, where $0 \le f \le \lfloor \frac{n}{2} \rfloor$ and λ is a multipartition of $n-2f$, which the theory of cellular algebras says are either zero or absolutely irreducible. Now, the cyclotomic Wenzl algebra $\mathscr{W}_{r,n}(\mathbf{u})$ is filtered by two sided ideals with the degenerate Hecke algebras $\mathscr{H}_{r,n-2f}(\mathbf{u})$ of type $G(r,1,n-2f)$ appearing as the successive quotients for $0 \le f \le \lfloor \frac{n}{2} \rfloor$. In section 6 we show that the algebras $\mathscr{H}_{r,m}(\mathbf{u})$ are also cellular (in fact, this is the key to proving Theorem B); as a consequence, the irreducible $\mathscr{H}_{r,m}(\mathbf{u})$–modules are the non–zero modules D^λ, where λ is a multipartition of m. Combining these facts we obtain the following classification of the irreducible $\mathscr{W}_{r,n}(\mathbf{u})$–modules in terms of the irreducible $\mathscr{H}_{r,n-2f}(\mathbf{u})$–modules, for $0 \le f \le \lfloor \frac{n}{2} \rfloor$.

Theorem C. *Suppose that R is a field with $2 \nmid \operatorname{char} R$, that Ω is $\mathbf{u}$–admissible and that $\omega_0 \ne 0$. Then $\{\, D^{(f,\lambda)} \mid 0 \le f \le \lfloor \frac{n}{2} \rfloor, \lambda \vdash n-2f \text{ and } D^\lambda \ne 0 \,\}$ is a complete set of pairwise non–isomorphic irreducible $\mathscr{W}_{r,n}(\mathbf{u})$–modules.*

As an application we give necessary and sufficient conditions for $\mathscr{W}_{r,n}(\mathbf{u})$ to be quasi–hereditary when R is a field and $\omega_0 \ne 0$.

2. Affine and cyclotomic Wenzl algebras

In [Naz96], Nazarov introduced an affine analogue of the Brauer algebra which he called the (degenerate) affine Wenzl algebra. Our main object of interest in this paper are certain "cyclotomic" quotients of Nazarov's algebra. In this section we define these algebras and prove some elementary results about them.

Fix a positive integer n and a commutative ring R with multiplicative identity 1_R. *Throughout this paper we will assume that* 2 *is invertible in* R.

Definition 2.1 (Nazarov [Naz96, §4])**.** Fix $\Omega = \{\,\omega_a \mid a \ge 0\,\} \subseteq R$. The degenerate affine Wenzl algebra $\mathscr{W}_n^{\mathrm{aff}} = \mathscr{W}_n^{\mathrm{aff}}(\Omega)$ is the unital associative R–algebra with generators $\{\,S_i, E_i, X_j \mid 1 \le i < n \text{ and } 1 \le j \le n\,\}$ and relations

a) (Involutions)
$S_i^2 = 1$, for $1 \le i < n$.

b) (Affine braid relations)
- (*i*) $S_iS_j = S_jS_i$ if $|i-j| > 1$,
- (*ii*) $S_iS_{i+1}S_i = S_{i+1}S_iS_{i+1}$, for $1 \le i < n-1$,
- (*iii*) $S_iX_j = X_jS_i$ if $j \ne i, i+1$.

c) (Idempotent relations)
$E_i^2 = \omega_0 E_i$, for $1 \le i < n$.

d) (Commutation relations)
- (*i*) $S_iE_j = E_jS_i$, if $|i-j| > 1$,
- (*ii*) $E_iE_j = E_jE_i$, if $|i-j| > 1$,
- (*iii*) $E_iX_j = X_jE_i$, if $j \ne i, i+1$,
- (*iv*) $X_iX_j = X_jX_i$, for $1 \le i, j \le n$.

e) (Skein relations)
$S_iX_i - X_{i+1}S_i = E_i - 1$ and
$X_iS_i - S_iX_{i+1} = E_i - 1$, for $1 \le i < n$.

f) (Unwrapping relations)
$E_1X_1^aE_1 = \omega_aE_1$, for $a > 0$.

g) (Tangle relations)
- (*i*) $E_iS_i = E_i = S_iE_i$, for $1 \le i \le n-1$,
- (*ii*) $S_iE_{i+1}E_i = S_{i+1}E_i$, for $1 \le i \le n-2$,
- (*iii*) $E_{i+1}E_iS_{i+1} = E_{i+1}S_i$, for $1 \le i \le n-2$.

h) (Untwisting relations)
$E_{i+1}E_iE_{i+1} = E_{i+1}$ and
$E_iE_{i+1}E_i = E_i$, for $1 \le i \le n-2$.

i) (Anti–symmetry relations)
$E_i(X_i + X_{i+1}) = 0$ and
$(X_i + X_{i+1})E_i = 0$, for $1 \le i < n$.

Our definition of $\mathscr{W}_n^{\mathrm{aff}}$ differs from Nazarov's in two respects. First, Nazarov considers only the special case when $R = \mathbb{C}$; however, as we will indicate, most of the arguments that we need from [Naz96] go through without change when R is an arbitrary ring. More significantly, Nazarov considers a more general algebra which is generated by the elements $\{\,S_i, E_i, X_j, \hat\omega_a \mid 1 \le i < n, 1 \le j \le n \text{ and } a \ge 0\,\}$ such that the $\hat\omega_a$ are central and the remaining generators satisfy the relations above. For our purposes it is more natural to define the elements ω_a to be elements of R because without this assumption the cyclotomic quotients of $\mathscr{W}_n^{\mathrm{aff}}$ would not be finite dimensional.

Note that $E_iE_{i+1}S_i = E_iE_{i+1}E_iS_{i+1} = E_iS_{i+1}$ and $S_{i+1}E_iE_{i+1} = S_iE_{i+1}E_iE_{i+1} = S_iE_{i+1}$. Thus a quick inspection of the defining relations shows that $\mathscr{W}_n^{\mathrm{aff}}$ has the following useful involution.

2.2. *There is a unique anti–isomorphism* $*\colon \mathscr{W}_n^{\mathrm{aff}} \longrightarrow \mathscr{W}_n^{\mathrm{aff}}$ *such that*

$$S_i^* = S_i, \qquad E_i^* = E_i \quad \text{and} \quad X_j^* = X_j,$$

for all $1 \le i < n$ *and all* $1 \le j \le n$. *Moreover,* $*$ *is an involution.*

Using the defining relations it is not hard to see that $\mathscr{W}_n^{\mathrm{aff}}$ is generated by the elements $S_1, \dots, S_{n-1}, E_1, X_1$. There is no real advantage, however, to using this smaller set of generators as the corresponding relations are more complicated.

Lemma 2.3 (cf. [Naz96, (2.6)])**.** *Suppose that* $1 \le i < n$ *and that* $a \ge 1$. *Then*

$$S_iX_i^a = X_{i+1}^aS_i + \sum_{b=1}^{a} X_{i+1}^{b-1}(E_i - 1)X_i^{a-b}.$$

Proof. We argue by induction on a. When $a=1$ this is relation 2.1(e) . If $a\ge 1$ then, by induction, we have

$$\begin{aligned} S_iX_i^{a+1} &= S_iX_i^aX_i = \Big\{X_{i+1}^aS_i + \sum_{b=1}^a X_{i+1}^{b-1}(E_i-1)X_i^{a-b}\Big\}X_i \\ &= X_{i+1}^aS_iX_i + \sum_{b=1}^a X_{i+1}^{b-1}(E_i-1)X_i^{a+1-b}. \end{aligned}$$

Now, by the skein relation 2.1(e), $S_iX_i = X_{i+1}S_i + E_i - 1$, so

$$\begin{aligned} S_iX_i^{a+1} &= X_{i+1}^a\big(X_{i+1}S_i + E_i - 1\big) + \sum_{b=1}^a X_{i+1}^{b-1}(E_i-1)X_i^{a+1-b} \\ &= X_{i+1}^{a+1}S_i + \sum_{b=1}^{a+1} X_{i+1}^{b-1}(E_i-1)X_i^{a+1-b}, \end{aligned}$$

as required. □

Corollary 2.4. *Suppose that $a\ge 0$. Then*

$$\omega_{2a+1}E_1 = \frac{1}{2}\Big\{-\omega_{2a} + \sum_{b=1}^{2a+1}(-1)^{b-1}\omega_{b-1}\omega_{2a+1-b}\Big\}E_1.$$

Proof. Take $i=1$ and multiply the equation in Lemma 2.3 on the left and right by E_1. Since $S_1E_1 = E_1 = E_1S_1$, this gives

$$E_1X_1^aE_1 = E_1X_2^aE_1 + \sum_{b=1}^a E_1X_2^{b-1}(E_1-1)X_1^{a-b}E_1.$$

Since $E_1X_1^cE_1 = \omega_cE_1$, $E_1(X_1+X_2)=0$ and $X_1X_2 = X_2X_1$ we can rewrite this equation as

$$\begin{aligned} \omega_aE_1 &= (-1)^a\omega_aE_1 + \sum_{b=1}^a(-1)^{b-1}E_1X_1^{b-1}(E_1-1)X_1^{a-b}E_1 \\ &= (-1)^a\omega_aE_1 + \sum_{b=1}^a(-1)^{b-1}\big(E_1X_1^{b-1}E_1X_1^{a-b}E_1 - E_1X_1^{a-1}E_1\big). \\ &= (-1)^a\omega_aE_1 + \sum_{b=1}^a(-1)^{b-1}(\omega_{b-1}\omega_{a-b} - \omega_{a-1})E_1. \\ &= (-1)^a\omega_aE_1 + \sum_{b=1}^a(-1)^{b-1}\omega_{b-1}\omega_{a-b}E_1 + \sum_{b=1}^a(-1)^b\omega_{a-1}E_1. \end{aligned}$$

Setting $a = 2a'+1$ proves the Corollary. □

If we assume that $E_1\ne 0$ in $\mathscr{W}_n^{\mathrm{aff}}$ and that $\mathscr{W}_n^{\mathrm{aff}}$ is torsion free then this result says that the ω_a, for a odd, are determined by the ω_b, for b even.

Remark 2.5. *If $a>0$ then the proof of the Corollary also gives the identity*

$$0 = \Big\{\sum_{b=1}^{2a}(-1)^{b-1}\omega_{b-1}\omega_{2a-b}\Big\}E_1.$$

However, this relation holds automatically because

$$\begin{aligned}\sum_{b=1}^{2a}(-1)^{b-1}\omega_{b-1}\omega_{2a-b} &= \sum_{b=1}^{a}(-1)^{b-1}\omega_{b-1}\omega_{2a-b} + \sum_{b=a+1}^{2a}(-1)^{b-1}\omega_{b-1}\omega_{2a-b}\\ &= \sum_{b=1}^{a}(-1)^{b-1}\omega_{b-1}\omega_{2a-b} + \sum_{b'=1}^{a}(-1)^{2a-b'}\omega_{2a-b'}\omega_{b'-1}\\ &= 0.\end{aligned}$$

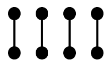
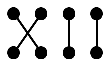
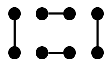
Before we define the cyclotomic quotients of $\mathscr{W}_n^{\mathrm{aff}}$, which are the main objects of study in this paper, we recall some standard definitions and notation from the theory of Brauer algebras and some of Nazarov's results.

A Brauer diagram on the $2n$ vertices $\{1,\dots,n,\overline{1},\dots,\overline{n}\}$ is a graph with n edges such that each vertex lies on a (unique) edge. Equivalently, a Brauer diagram is a partitioning of $\{1,\dots,n,\overline{1}\dots,\overline{n}\}$ into n two element subsets. Let $\mathcal{B}(n)$ be the set of all Brauer diagrams on $\{1,\dots,n,\overline{1},\dots,\overline{n}\}$. Then $\#\mathcal{B}(n) = 2^n(2n-1)!!$.

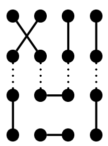
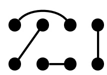
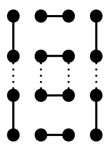
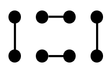
Let $\gamma \in \mathcal{B}(n)$ be a Brauer diagram. A vertical edge in γ is any edge of the form $\{m,\overline{m}\}$, where $1 \le m \le n$. Horizontal edges are edges of the form $\{m,p\}$, or $\{\overline{m},\overline{p}\}$, where $1 \le m < p \le n$.

For $i = 1,\dots,n-1$ let $\gamma(i,i+1)$ be the Brauer diagram with edges $\{i,\overline{i+1}\}$, $\{i+1,\overline{i}\}$ and all other edges being vertical. Similarly, let γ_i be the Brauer diagram with edges $\{i,i+1\}$, $\{\overline{i},\overline{i+1}\}$, and with all other edges being vertical. We set $s_i = b_{\gamma(i,i+1)}$ and $e_i = b_{\gamma_i}$. We also let γ_e be the graph with edges $\{\,\{i,\overline{i}\} \mid 1 \le i \le n\,\}$.

Brauer diagrams can be represented diagrammatically as in the following examples. The vertices in the first rows are labelled from left to right as 1 to 4, and the vertices in the second row are labelled $\overline{1}$ to $\overline{4}$.

γ_e =  , $\gamma(1,2)$ =  , and γ_2 =  .

Given two Brauer diagrams $\gamma,\gamma' \in \mathcal{B}(n)$ we define their product to be the diagram $\gamma \bullet \gamma'$ which is obtained by identifying vertex $\overline{i}$ in γ with vertex i in γ', for $1 \le i \le n$. Let $\ell(\gamma,\gamma')$ be the number of loops in the graph $\gamma \bullet \gamma'$ and let $\gamma \circ \gamma'$ be the Brauer diagram obtained by deleting these loops. The following pictures give two examples of the multiplication $\gamma \circ \gamma'$ of diagrams.

 =  and  = 

In the first example $\gamma = \gamma(1,2)$, $\gamma' = \gamma_2$ and $\ell(\gamma,\gamma') = 0$. In the second example $\gamma = \gamma' = \gamma_2$ and $\ell(\gamma,\gamma') = 1$.

Recall that R is a commutative ring.

Definition 2.6 (Brauer [Bra37])**.** Suppose that $\omega \in R$. The Brauer algebra $\mathscr{B}_n(\omega)$, with parameter ω, is the R–algebra which is free as an R–module with basis $\{\, b_\gamma \mid \gamma \in \mathcal{B}(n)\,\}$ and with multiplication determined by

$$b_\gamma b_{\gamma'} = \omega^{\ell(\gamma,\gamma')} b_{\gamma\circ\gamma'},$$

for $\gamma,\gamma' \in \mathcal{B}(n)$.

It is easy to see that $\mathscr{B}_n(\omega)$ is an associative algebra with identity b_{γ_e}. We abuse notation and sometimes write $1 = b_{\gamma_e}$.

The second example above indicates that $e_i^2 = \omega e_i$, for $1 \le i < n$. Similarly, $s_i^2 = 1$, for $1 \le i < n$.

Let $\mathfrak{S}_n$ be the symmetric group on n letters. To each permutation $w \in \mathfrak{S}_n$ we associate the Brauer diagram $\gamma(w)$ which has edges $\{\{i, \overline{w(i)}\} \mid \text{for } 1 \le i \le n\}$. Notice that if $w = (i, i+1)$ then this is consistent with the notation introduced above for the elements $s_i = b_{\gamma(i,i+1)} \in \mathscr{B}_n(\omega)$.

The diagrams $\{\gamma(w) \mid w \in \mathfrak{S}_n\}$ are precisely the Brauer diagrams which do not have any horizontal edges. It is easy to see that the map $w \mapsto b_{\gamma(w)}$ induces an algebra embedding of the group ring $R\mathfrak{S}_n$ of $\mathfrak{S}_n$ into $\mathscr{B}_n(\omega)$. In this way, $R\mathfrak{S}_n$ can be considered as a subalgebra of $\mathscr{B}_n(\omega)$.

There is a well–known presentation of $\mathscr{B}_n(\omega)$, which we now describe. When $R = \mathbb{C}$ this result is proved in [BW89].

Proposition 2.7. *Suppose that R is a commutative ring. The Brauer algebra $\mathscr{B}_n(\omega)$ is generated by the elements $s_1, \dots, s_{n-1}, e_1, \dots, e_{n-1}$ subject to the relations*

$$\begin{array}{ccc}
s_i^2 = 1, & e_i^2 = \omega e_i, & s_i e_i = e_i s_i = e_i, \\
s_i s_j = s_j s_i, & s_i e_j = e_j s_i, & e_i e_j = e_j e_i, \\
s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}, & e_k e_{k+1} e_k = e_k, & e_{k+1} e_k e_{k+1} = e_{k+1}, \\
s_k e_{k+1} e_k = s_{k+1} e_k, & e_{k+1} e_k s_{k+1} = e_{k+1} s_k, &
\end{array}$$

where $1 \le i, j < n$, with $|i - j| > 1$, and $1 \le k < n - 1$

Sketch of proof. Let $B_n(\omega)$ be the algebra with the presentation above. It is easy to see that $\mathscr{B}_n(\omega)$ is generated by $\{b_{\gamma_i}, b_{\gamma(i,i+1)} \mid 1 \le i < n\}$ and that these elements satisfy all of the relations above. Hence, there is a well–defined algebra homomorphism $\theta : B_n(\omega) \longrightarrow \mathscr{B}_n(\omega)$ which is determined by $\theta(s_i) = b_{\gamma(i,i+1)}$ and $\theta(e_i) = b_{\gamma_i}$. As θ is surjective, $\mathscr{B}_n(\omega) \cong B_n(\omega)/\ker\theta$. It is not hard to show that $B_n(\omega)$ is spanned by words in the generators which are naturally indexed by the Brauer diagrams $\mathcal{B}(n)$; see, for example, [Wen88, Prop. 2.1(a)]. It follows that $\operatorname{Ker}\theta = 0$ since $\mathscr{B}_n(\omega)$ is R–free, so that $B_n(\omega) \cong \mathscr{B}_n(\omega)$ as required. □

Let $s_{ij} = b_{\gamma(i,j)}$, and let $e_{ij} = b_{\gamma_{ij}}$ where γ_{ij} is the Brauer diagram with edges $\{i, j\}$, $\{\bar{i}, \bar{j}\}$ and $\{k, \bar{k}\}$, for $k \ne i, j$.

Corollary 2.8 (Nazarov [Naz96, (2.2)])**.** *Suppose that $\omega \in R$ and let $\Omega = \{\omega_a \mid a \ge 0\}$, where $\omega_a = \omega\left(\frac{\omega-1}{2}\right)^a$, for $a \ge 0$. Then there is a surjective algebra homomorphism $\pi : \mathscr{W}_n^{\mathrm{aff}}(\Omega) \longrightarrow \mathscr{B}_n(\omega)$ which is determined by*

$$\pi(S_i) = s_i, \quad \pi(E_i) = e_i, \quad \text{and} \quad \pi(X_j) = \frac{\omega - 1}{2} + \sum_{k=1}^{j-1} s_{kj} - e_{kj},$$

for $1 \le i < n$ and $1 \le j \le n$. Moreover, $\ker\pi = \langle X_1 - (\frac{\omega-1}{2})\rangle$, so that

$$\mathscr{W}_n^{\mathrm{aff}}(\Omega)/\big\langle X_1 - (\tfrac{\omega-1}{2})\big\rangle \cong \mathscr{B}_n(\omega).$$

Notice, in particular, that $\pi(X_1) = \frac{\omega-1}{2}$. To prove this result it is enough to show that the elements $\pi(X_j)$, for $1 \le j \le n$, satisfy the relations in $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$. For these calculations see [Naz96, Lemma 2.1 and Proposition 2.3].

Fix a Brauer diagram $\gamma \in \mathcal{B}$. By Proposition 2.7 we can write b_γ as a word in the generators $s_1, \dots, s_{n-1}, e_1, \dots, e_{n-1}$. Fix such a word for b_γ and let $B_\gamma \in \mathscr{W}_n^{\mathrm{aff}}(\Omega)$ be the corresponding word in the generators $S_1, \dots, S_{n-1}, E_1, \dots, E_{n-1}$. Then $\pi(B_\gamma) = b_\gamma$.

Given $\alpha, \beta \in \mathbb{N}_0^n$ and $\gamma \in \mathcal{B}(n)$ write

$$X^\alpha B_\gamma X^\beta = X_1^{\alpha_1} \dots X_n^{\alpha_n} B_\gamma X_1^{\beta_1} \dots X_n^{\beta_n}.$$

We want to use these monomials to give a basis of $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$. The anti–symmetry relations $E_i(X_i + X_{i+1}) = 0$, for $1 \le i < n$, show that the set of all monomials is not linearly independent. In Theorem 2.12 below we will show that the following monomials are linearly independent.

Definition 2.9. Suppose that $\alpha,\beta\in\mathbb{N}_0^n$ and $\gamma\in\mathcal{B}(n)$. A monomial $X^\alpha B_\gamma X^\beta$ in $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$ is regular if

a) $\alpha_r=0$ whenever r is the right endpoint of a horizontal edge in the top row of γ.

b) if $\beta_l\ne 0$ then l is the left endpoint of a horizontal edge in the bottom row of γ.

We can view a regular monomial $X^\alpha B_\gamma X^\beta$ as a Brauer diagram if we colour the horizontal and vertical edges with the non–negative integers using α and β.

Following Corollary 2.4 we also make the following definition. (Recall that we are assuming that 2 is a unit in R.)

Definition 2.10. Let $\Omega=\{\,\omega_a\in R\mid a\ge 0\,\}$. Then Ω is admissible if

$$\omega_{2a+1}=\frac{1}{2}\Big\{-\omega_{2a}+\sum_{b=1}^{2a+1}(-1)^{b-1}\omega_{b-1}\omega_{2a+1-b}\Big\},$$

for all $a\ge 0$

Remark 2.11. *Let y be an indeterminate and consider the generating series $\widetilde{W}_1(y)=\sum_{a\ge 0}\omega_a y^{-a}$. Then the condition for Ω to be admissible can be written as*

$$\big(\widetilde{W}_1(y)+y-\frac{1}{2}\big)\big(\widetilde{W}_1(-y)-y-\frac{1}{2}\big)=(\frac{1}{2}-y)(\frac{1}{2}+y).$$

Similar generating functions play an important role in section 4.

Theorem 2.12 (Nazarov [Naz96, Theorem 4.6])**.** *Suppose R is a commutative ring in which 2 is a unit and that $\Omega=\{\,\omega_a\in R\mid a\ge 0\,\}$ is admissible. Then $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$ is free as an R–module with basis $\{\,X^\alpha B_\gamma X^\beta\mid \alpha,\beta\in\mathbb{N}_0^n,\gamma\in\mathcal{B}(n),$ and $X^\alpha B_\gamma X^\beta$ is regular$\}$.*

Sketch of proof. We have defined the elements of Ω to be scalars, but Nazarov [Naz96] works with a larger algebra $\widehat{\mathscr{W}_n^{\mathrm{aff}}}(\widehat{\Omega})$ generated by elements S_i, E_i, X_j, for $1\le i<n$ and $1\le j\le n$, and $\widehat{\Omega}=\{\,\widehat{\omega}_a\mid a\ge 0\,\}$ where these generators satisfy the same relations as the corresponding generators of $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$ except that the elements of Ω are central elements of $\widehat{\mathscr{W}_n^{\mathrm{aff}}}(\widehat{\Omega})$, rather than scalars. Hence, $\mathscr{W}_n^{\mathrm{aff}}(\Omega)\cong\widehat{\mathscr{W}_n^{\mathrm{aff}}}(\widehat{\Omega})/I$, where I is the two sided ideal of $\widehat{\mathscr{W}_n^{\mathrm{aff}}}(\widehat{\Omega})$ generated by the elements $\{\,\widehat{\omega}_a-\omega_a\mid a\ge 0\,\}$.

Nazarov puts a grading on $\widehat{\mathscr{W}_n^{\mathrm{aff}}}(\widehat{\Omega})$ by setting $\deg S_i=\deg E_i=\deg\widehat{\omega}_a=0$ and $\deg X_i=1$. To prove the result it is enough to work with the associated graded algebra $gr(\widehat{\mathscr{W}_n^{\mathrm{aff}}}(\widehat{\Omega}))$, where the grading is that induced by the degree function. The arguments of Lemma 4.4 and Lemma 4.5 from [Naz96] go through without change for an arbitrary ring. $\widehat{\mathscr{W}_n^{\mathrm{aff}}}(\widehat{\Omega})$ is spanned by

$$\Big\{\;X^\alpha B_\gamma X^\beta\widehat{\omega}_2^{h_2}\widehat{\omega}_4^{h_4}\dots\;\Big|\;\begin{matrix}\alpha,\beta\in\mathbb{N}_0^n,\ \gamma\in\mathcal{B}(n),\ h_{2i}\ge 0,\text{ for }i\ge 1,\\ \text{with only finitely many }h_{2i}\ne 0\end{matrix}\Big\},$$

where the monomials $X^\alpha B_\gamma X^\beta$ are all regular (see [Naz96, Theorem 4.6]). This implies that the regular monomials span $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$ for any ring R.

To complete the proof we first consider the case where the elements of Ω' are indeterminates over $\mathbb{Z}$ and we consider the affine Wenzl algebras defined over the field $\mathbb{C}(\Omega')$ and over the ring $\mathbb{Z}[\Omega']$. We write $\mathscr{W}_{R,n}^{\mathrm{aff}}(\Omega')=\mathscr{W}_n^{\mathrm{aff}}(\Omega')$ to emphasize that $\mathscr{W}_n^{\mathrm{aff}}(\Omega')$ is defined over the ring R.

Using Nazarov's algebra $\widehat{\mathscr{W}_n^{\mathrm{aff}}}(\widehat{\Omega}')$ and arguing as above, it follows from [Naz96, Lemma 4.8] that the set of regular monomials are linearly independent when $R = \mathbb{C}(\Omega')$. By the last paragraph, the regular monomials span $\mathscr{W}^{\mathrm{aff}}_{\mathbb{Z}[\Omega'],n}(\Omega')$. It follows, therefore, that $\mathscr{W}^{\mathrm{aff}}_{\mathbb{Z}[\Omega'],n}(\Omega')$ is free as a $\mathbb{Z}[\Omega']$–module and has basis the set of regular monomials. Hence, by a specialization argument, if R is arbitary ring R and $\Omega \subseteq R$ then

$$\mathscr{W}^{\mathrm{aff}}_{R,q}(\Omega) \cong \mathscr{W}^{\mathrm{aff}}_{\mathbb{Z}[\Omega'],n}(\Omega') \otimes_{\mathbb{Z}[\Omega']} R,$$

where we consider R as a $\mathbb{Z}[\Omega']$–module by letting $\omega'_a \in \Omega'$ act on R as multiplication by $\omega_a \in \Omega$, for $a \geq 0$. Hence, $\mathscr{W}^{\mathrm{aff}}_{R,n}(\Omega)$ is free as an R–module with basis the set of regular monomials as claimed. □

We are now ready to define the cyclotomic Wenzl algebras. We assune henceforth that Ω is admissible.

Definition 2.13. Fix an integer $r \geq 1$ and $\mathbf{u} = (u_1, \dots, u_r) \in R^r$. The cyclotomic Wenzl algebra $\mathscr{W}_{r,n} = \mathscr{W}_{r,n}(\mathbf{u})$ is the R–algebra $\mathscr{W}_n^{\mathrm{aff}}(\Omega)/\langle (X_1 - u_1) \dots (X_1 - u_r)\rangle$.

We should write $\mathscr{W}_{r,n}(\mathbf{u}, \Omega)$, however, in section 3 we will restrict to the case where Ω is $\mathbf{u}$–admissible (Definition 3.5), which implies that ω_a is determined by $\mathbf{u}$, for $a \geq 0$. For this reason we omit Ω from the notation for $\mathscr{W}_{r,n}(\mathbf{u})$.

By Corollary 2.8 the Brauer algebras $\mathscr{B}_n(\omega)$ are a special case of the cyclotomic Wenzl algebras corresponding to $r = 1$ and $\Omega = \{\, \omega(\frac{\omega-1}{2})^a \mid a \geq 0 \,\}$.

By definition there is a surjection $\pi_{r,n} : \mathscr{W}_n^{\mathrm{aff}}(\Omega) \longrightarrow \mathscr{W}_{r,n}(\mathbf{u})$. Abusing notation, we write $S_i = \pi_{r,n}(S_i)$, $E_i = \pi_{r,n}(E_i)$, $X_j = \pi_{r,n}(X_j)$, and $B_\gamma = \pi_{r,n}(B_\gamma)$ for $1 \leq i < n$, $1 \leq j \leq n$ and $\gamma \in \mathcal{B}(n)$.

Notice that because $(X_1 - u_1) \dots (X_1 - u_r) = 0$ in $\mathscr{W}_{r,n}(\mathbf{u})$ the cyclotomic Wenzl algebras have only r unwrapping relations; that is, we only need to impose the relations $E_1 X_1^a E_1 = \omega_a E_1$, for $0 \leq a \leq r-1$.

Every $\mathscr{W}_{r,n}(\mathbf{u})$–module can be considered as a $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$–module by inflation along the surjection $\pi_{r,n} : \mathscr{W}_n^{\mathrm{aff}}(\Omega) \longrightarrow \mathscr{W}_{r,n}(\mathbf{u})$. In particular, every irreducible $\mathscr{W}_{r,n}(\mathbf{u})$–module is also an irreducible $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$–module. This result has the following converse.

Corollary 2.14. *Suppose that R is an algebraically closed field and that M is a finite dimensional irreducible $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$–module. Then M can also be considered as an irreducible module for some cyclotomic Wenzl algebra $\mathscr{W}_{r,n}(\mathbf{u})$.*

Proof. Let $c_M(X)$ be the characteristic polynomial for the action of X_1 on M. Since R is algebraically closed $c_M(X) = (X - u_1) \dots (X - u_r)$, for some $u_s \in R$. Hence, $(X_1 - u_1) \dots (X_1 - u_r)$ acts as zero on M, so that M is an irreducible representation for $\mathscr{W}_{r,n}(\mathbf{u})$, where $\mathbf{u} = (u_1, \dots, u_r)$. □

In practice this result is not very useful because most of the results in this paper require that Ω be $\mathbf{u}$–admissible and it is unlikely that Ω will be $\mathbf{u}$–admissible for all of the parameters $\mathbf{u}$ that arise in this way.

For our first result about the cyclotomic Wenzl algebras we prove the easy half of Theorem A. That is, we show that $\mathscr{W}_{r,n}(\mathbf{u})$ is spanned by $r^n(2n-1)!!$ elements.

Definition 2.15. Suppose that $\alpha, \beta \in \mathbb{N}_0^n$ and $\gamma \in \mathcal{B}(n)$.

a) The monomial $X^\alpha B_\gamma X^\beta$ in $\mathscr{W}_{r,n}(\mathbf{u})$ is regular if $X^\alpha B_\gamma X^\beta$ is a regular monomial in $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$.

b) The monomial $X^\alpha B_\gamma X^\beta$ in $\mathscr{W}_{r,n}(\mathbf{u})$ is r–regular if it is regular and $0 \leq \alpha_i, \beta_i < r$, for all $1 \leq i \leq n$.

Proposition 2.16. *The cyclotomic Wenzl algebra $\mathscr{W}_{r,n}(\mathbf{u})$ is spanned by the set of r–regular monomials $\{X^\alpha B_\gamma X^\beta\}$. In particular, if R is a field then*

$$\dim_R \mathscr{W}_{r,n}(\mathbf{u}) \le r^n(2n-1)!!.$$

Proof. By Theorem 2.12, and the definitions, $\mathscr{W}_{r,n}(\mathbf{u})$ is spanned by the regular monomials in $\mathscr{W}_{r,n}(\mathbf{u})$. As in the proof of Theorem 2.12, we put a grading on $\mathscr{W}_{r,n}(\mathbf{u})$. Then in the associated graded algebra, $gr\,\mathscr{W}_{r,n}(\mathbf{u})$, we have the relation $(X_i-u_1)\cdots(X_i-u_r)=0$. We claim that the regular monomial $X^\alpha B_\gamma X^\beta$ can be written as a linear combination of r-regular monomials. When $a=|\alpha|+|\beta|=0$, there is nothing to prove. If $a>0$ then by subtracting a linear combination of r–regular monomials from $X^\alpha B_\gamma X^\beta$ we obtain a linear combination of regular elements of smaller degree. The claim now follows by induction.

Finally, a counting argument shows that the number of r–regular monomials is equal to $r^n(2n-1)!!$. Therefore, if R is a field then $\dim_R \mathscr{W}_{r,n}(\mathbf{u}) \le r^n(2n-1)!!$. □

The degenerate Hecke algebra $\mathscr{H}_{r,n}(\mathbf{u})$ of type $G(r,1,n)$ is the unital associative R–algebra with generators $T_1,\dots,T_{n-1},Y_1,\dots,Y_n$ and relations

$$\begin{aligned}
(Y_1-u_1)\dots(Y_1-u_r)&=0, & T_i^2&=1,\\
T_iT_j&=T_jT_i, & Y_iY_k&=Y_kY_i,\\
T_iY_i-Y_{i+1}T_i&=-1, & Y_iT_i-T_iY_{i+1}&=-1,\\
T_jT_{j+1}T_j&=T_{j+1}T_jT_{j+1},
\end{aligned}$$

for $1\le i<n$, $1\le j<n-1$ with $|i-j|>1$, and $1\le k\le n$. Therefore there is a surjective algebra homomorphism $\mathscr{W}_{r,n}(\mathbf{u}) \longrightarrow \mathscr{H}_{r,n}(\mathbf{u})$ determined by

$$S_i\mapsto T_i,\quad E_i\mapsto 0,\quad\text{and}\quad X_j\mapsto Y_j,$$

for $1\le i<n$ and $1\le j\le n$. (In fact, a special case of Proposition 7.2 below shows that $\mathscr{H}_{r,n}(\mathbf{u})\cong \mathscr{W}_{r,n}(\mathbf{u})/\langle E_1\rangle$.) Consequently, every irreducible $\mathscr{H}_{r,n}(\mathbf{u})$–module can be considered as an irreducible $\mathscr{W}_{r,n}(\mathbf{u})$–module via inflation. These irreducible modules are precisely the irreducibles upon which E_i acts as zero. We record this fact for future use.

Corollary 2.17. *Suppose that R is a field and that M is an irreducible $\mathscr{W}_{r,n}(\mathbf{u})$–module which is annihilated by some E_i. Then M is an irreducible $\mathscr{H}_{r,n}(\mathbf{u})$–module.*

Proof. As $E_{i+1}=S_iS_{i+1}E_iS_{i+1}S_i$ and S_j is invertible for all j, the two–sided ideal of $\mathscr{W}_{r,n}(\mathbf{u})$ generated by E_1 is the same as the two–sided ideal generated by E_i, for $1\le i<n$. The result now follows from the remarks above. □

Recall that the degenerate affine Hecke algebra is a finitely generated module over its center (see, for example, [Kle05]), which is the ring of the symmetric polynomials in $Y_1,\dots,Y_n$. This fact, together with Dixmier's version of Schur's lemma, implies that all of the irreducible modules of the degenerate affine Hecke algebra are finite dimensional. In contrast, *the affine Wenzl algebra is not finitely generated over its center.* To see this, we give an example of an infinite dimensional irreducible $\mathscr{W}_2^{\mathrm{aff}}(\Omega)$-module.

Example 2.18. Suppose that Ω is admissible and consider $V=\oplus_{n\ge0}Rv_n$. Define an action of $\mathscr{W}_2^{\mathrm{aff}}(\Omega)$ on V by $Ev_n=\omega_nv_0$, $X_1v_n=v_{n+1}$, $X_2v_n=-v_{n+1}$ and

$$Sv_n=(-1)^nv_n-\varepsilon v_{n-1}+\sum_{k=0}^{n-1}(-1)^k\omega_{n-k-1}v_k,$$

where $\varepsilon=1$, if $n\equiv1 \pmod 2$, and $\varepsilon=0$, otherwise. All of the defining relations except for the relation $S^2=1$ are easy to check. As S^2 commutes with X_1, $S^2v_0=v_0$ and $X_1v_n=v_{n+1}$, we have that S^2 acts as the identity on V.

Now we show that V is irreducible. Let W be an irreducible $\mathscr{W}_2^{\mathrm{aff}}(\Omega)$-submodule of V. Suppose that $EW = 0$. Then W may be viewed as an irreducible $\mathscr{H}_2^{\mathrm{aff}}$-module, which implies that W is finite dimensional. In particular, W contains an eigenvector of X_1. This is impossible, as V does not have such an eigenvector. We have $EW \neq 0$, which implies that $v_0 \in W$. Hence $W = V$.

In light of this example, we restrict our attention to finite dimensional $\mathscr{W}_n^{\mathrm{aff}}(\Omega)$–modules in what follows.

3. Restrictions on Ω and the irreducible representations of $\mathscr{W}_{r,2}$

In this section we explicitly compute the (possible) irreducible representations of the cyclotomic Wenzl algebras $\mathscr{W}_{r,2}(\mathbf{u})$. As a consequence we find a set of conditions on the parameter set Ω which ensure that $\mathscr{W}_{r,2}(\mathbf{u})$ has dimension $3r^2 = r^n(2n-1)!!\,|_{n=2}$ when R is a field. In the next section we will see that these conditions on Ω are exactly what we need for general n.

The cyclotomic Wenzl algebra $\mathscr{W}_{r,2}(\mathbf{u})$ is generated by S_1, E_1, X_1 and X_2. Throughout this section we suppose that R is an algebraically closed field and, for convenience, we set $S = S_1$ and $E = E_1$.

Proposition 3.1. *Suppose that M is an irreducible $\mathscr{W}_{r,2}(\mathbf{u})$–module such that $EM = 0$. Then either:*

a) $M = Rm$ is one dimensional and the action of $\mathscr{W}_{r,2}(\mathbf{u})$ is determined by

$$Sm = \varepsilon m, \quad Em = 0, \quad X_1 m = u_i m, \quad \textit{and} \quad X_2 m = (u_i + \varepsilon)m,$$

where $\varepsilon = \pm 1$ and $1 \leq i \leq r$. In particular, up to isomorphism, there are at most $2r$ such representations.

b) M is two dimensional and the action of $\mathscr{W}_{r,2}(\mathbf{u})$ is given by

$$S \mapsto \tfrac{1}{u_i - u_j}\begin{pmatrix} -1 & b \\ c & 1 \end{pmatrix}, \quad E \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_1 \mapsto \begin{pmatrix} u_i & 0 \\ 0 & u_j \end{pmatrix}, \quad \textit{and} \quad X_2 \mapsto \begin{pmatrix} u_j & 0 \\ 0 & u_i \end{pmatrix},$$

for some non–zero $b, c \in R$ such that $bc = (u_i - u_j)^2 - 1$, where $u_i \neq u_j$. Up to isomorphism there are at most $\binom{r}{2}$ such representations.

c) M is two dimensional and the action of $\mathscr{W}_{r,2}(\mathbf{u})$ is given by

$$S \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_1 \mapsto \begin{pmatrix} u_i & -1 \\ 0 & u_i \end{pmatrix}, \quad \textit{and} \quad X_2 \mapsto \begin{pmatrix} u_i & 1 \\ 0 & u_i \end{pmatrix}.$$

Up to isomorphism there are at most r such representations.

Proof. As noted in Corollary 2.17 M is an irreducible $\mathscr{H}_{r,2}(\mathbf{u})$–module. The result now follows from the representation theory of $\mathscr{H}_{r,2}(\mathbf{u})$: choose a simultaneous eigenvector m of $R[Y_1, Y_2]$. Then, because $\mathscr{H}_{r,2}(\mathbf{u}) = R[Y_1, Y_2] + T_1 R[Y_1, Y_2]$, if M is not one dimensional then it must be two dimensional. If this is the case, $\{m, Sm\}$ is a basis of M. Further, if the eigenvalues for the action Y_1 on M are distinct, then we can simultaneously diagonalize Y_1 and Y_2. All of our claims now follow. □

Note that since $\prod_{i=1}^r (Y_1 - u_i)$ acts as zero on M, case (c) can only arise if there exist $i \neq j$ with $u_i = u_j$. The irreducible representations of $\mathscr{W}_{r,2}(\mathbf{u})$ upon which E acts non–trivially take more effort to understand.

Proposition 3.2. *Let F be a field and suppose that $2 \nmid \operatorname{char} F$ and that $u_1, \dots, u_r$ are algebraically independent over F. Let $R = F(u_1, \dots, u_r)$ and let $\mathscr{W}_{r,2}(\mathbf{u})$ be the cyclotomic Wenzl algebra defined over R, where $\omega_0 \neq 0$. Then $\mathscr{W}_{r,2}(\mathbf{u})$ has a unique irreducible module M such that $EM \neq 0$. Moreover, if $d = \dim_R M$ then $d \leq r$ and there exists a basis $\{m_1, \dots, m_d\}$ of M and scalars $\{v_1, \dots, v_d\} \subseteq \{u_1, \dots, u_r\}$, with $v_i \neq v_j$ when $i \neq j$, such that for $1 \leq i \leq d$ the following hold:*

a) $X_1 m_i = v_i m_i$ and $X_2 m_i = -v_i m_i$,

b) $E m_i = \gamma_i (m_1 + \cdots + m_d)$ and *c)* $Sm_i = \dfrac{\gamma_i - 1}{2v_i} m_i + \sum_{j\neq i} \dfrac{\gamma_i}{v_i + v_j} m_j,$

where $\gamma_i = (2v_i - (-1)^d) \prod_{\substack{1\le j\le d\\ j\neq i}} \dfrac{v_i + v_j}{v_i - v_j}$. *Moreover,* $\omega_a = \sum_{j=1}^d v_j^a \gamma_j$, *for all* $a \geq 0$; *and, in particular,*

$$\omega_0 = \begin{cases} 2(v_1 + \cdots + v_d), & \text{if } d \text{ is even,}\\ 2(v_1 + \cdots + v_d) + 1, & \text{if } d \text{ is odd.}\end{cases}$$

Conversely, if $\omega_a = \sum_{j=1}^d v_j^a \gamma_j$, *for all* $a \geq 0$, *then (a)–(c) define a* $\mathscr{W}_{r,2}(\mathbf{u})$*–module* M *with* $EM \neq 0$.

Proof. Suppose that M is an irreducible $\mathscr{W}_{r,2}$–module such that $EM \neq 0$. Note that M is finite dimensional. Let $d = \dim_R M$. We first show that (a)–(c) hold.

To start with suppose that $d = 1$. Then E acts as multiplication by a scalar λ and, by assumption, $\lambda \neq 0$. Since $E^2 = \omega_0 E$ we must have $\lambda = \omega_0$. Further, S must act as the identity on M since $E = SE$. Suppose that X_1 acts as multiplication by v_1. Since $(X_1 - u_1)(X_1 - u_2)\cdots(X_1 - u_r) = 0$, $v_1 \in \{u_1, u_2, \cdots, u_r\}$, so $v_1 = u_i$ for some i. However, the relation $EX_1^k E = \omega_k E$ now forces $\omega_k = \omega_0 u_i^k$. On the other hand, $\omega_1 = \frac{1}{2}\omega_0(\omega_0 - 1)$, since $SX_1 - X_2 S = E - 1$, so that $2\omega_0 v_1 = \omega_0(\omega_0 - 1)$. Consequently, $\omega_0 = 2v_1 + 1 = \gamma_1$ as required.

Now suppose that $d \geq 2$ and that E does not act as zero on M. Since $u_1, \ldots, u_r$ are pairwise distinct, we can fix a basis $\{m_1, \ldots, m_d\}$ of M consisting of eigenvectors for X_1. Write $X_1 m_i = v_i m_i$, for some $v_i \in \{u_1, \cdots, u_r\}$.

Set $f := \frac{1}{\omega_0} E$. This is a non–zero idempotent and $fM \neq 0$ since $EM \neq 0$.

Fix an element $0 \neq m \in fM$. Then $Em = \omega_0 m$ and $Sm = m$ (since $SE = E$). As $0 = (X_1 + X_2)Em = (X_1 + X_2)\omega_0 m$, we have $(X_1 + X_2)m = 0$. However, $X_1 + X_2$ is central in $\mathscr{W}_{r,2}$, so $X_1 + X_2$ acts as a scalar on M by Schur's lemma. Hence, $X_2 m_i = -X_1 m_i = -v_i m_i$, for $i = 1, \ldots, d$, proving (a).

We claim that $\{m, X_1 m, \ldots, X_1^{d-1} m\}$ is a basis of M. To see this, for any $a \geq 0$ let M_a be the R–submodule of M spanned by $\{m, X_1 m, \ldots, X_1^a m\}$. Notice that M_a is closed under left multiplication by E since if $k \geq 0$ then

$$EX_1^k m = EX_1^k fm = \frac{1}{\omega_0} EX_1^k Em = \frac{\omega_k}{\omega_0} Em = \omega_k m.$$

Also, by Lemma 2.3,

$$\begin{aligned} SX_1^a m &= \big(X_2^a S + \sum_{b=1}^a X_2^{b-1}(E-1)X_1^{a-b}\big)m\\ &= X_2^a m + \sum_{b=1}^a \big(X_2^{b-1} E X_1^{a-b} E \frac{1}{\omega_0} m - X_1^{a-b} X_2^{b-1} m\big)\\ &= X_2^a m + \sum_{b=1}^a \big(\frac{\omega_{a-b}}{\omega_0} X_2^{b-1} Em - X_1^{a-b} X_2^{b-1} m\big)\\ &= (-1)^a X_1^a m + \sum_{b=1}^a \big((-1)^{b-1}\omega_{a-b} X_1^{b-1} m - (-1)^{b-1} X_1^{a-1} m\big) \end{aligned}$$

So, M_a is closed under multiplication by S. Choose $a \geq 0$ to be minimal such that $\{m, X_1 m, \ldots, X_1^{a+1} m\}$ is not linearly independent. Since $X_1^{a+1} m \in M_a$, M_a is closed under multiplication by X_1. Hence, $M_a = M$ since M is irreducible. By counting dimensions, $M = M_{d-1}$, proving the claim.

Next we show that $EM = Rm$. Suppose that $m' = \sum_{i=0}^{d-1} c_i X_1^i m \in EM$. Then

$$m' = \frac{1}{\omega_0} Em' = \frac{1}{\omega_0} \sum_{i=0}^{d-1} c_i E X_1^i m = \frac{1}{\omega_0^2} \sum_{i=0}^{d-1} c_i E X_1^i E m = \frac{1}{\omega_0^2} \Big(\sum_{i=0}^{d-1} c_i \omega_i \Big) m,$$

since $Ea = \omega_0 a$ whenever $a \in EM$. Hence, $EM = Rm$, as claimed.

Recall that we have fixed a basis $\{m_1, \dots, m_d\}$ of M. Write $m = \sum_{i=1}^d r_i m_i$, for some $r_i \in R$. Suppose that $r_i = 0$ for some i. Then

$$\prod_{\substack{1 \le j \le d \\ j \ne i}} (X_1 - v_j) \cdot m = 0.$$

This contradicts the linear independence of $\{m, X_1 m, \dots, X_1^{d-1} m\}$; hence, $r_i \ne 0$ for $i = 1, \dots, d$. By rescaling the m_i, if necessary, we can and do assume that $m = m_1 + \cdots + m_d$ in the following.

By the argument of the last paragraph, all of the eigenvalues $\{v_1, \dots, v_d\}$ of m must be distinct. This also shows that $d = \dim M \le r$ and that $\{v_1, \dots, v_d\}$ are algebraically independent (since we are assuming that $u_1, \dots, u_r$ are algebraically independent). In particular, v_i and $v_i + v_j$, for $i \ne j$, are invertible. So the formula in part (c) makes sense.

As $EM = Rm$, we can define elements $\gamma_i \in R$ by

$$Em_i = \gamma_i m = \gamma_i (m_1 + \cdots + m_d), \qquad \text{for } i = 1, \dots, d.$$

Write $Sm_i = \sum_{j=1}^d c_j^{(i)} m_j$. Then $X_1 S m_i - S X_2 m_i = (E-1) m_i$ reads

$$\sum_{j=1}^d c_j^{(i)} v_j m_j + v_i \big(\sum_{j=1}^d c_j^{(i)} m_j \big) = \gamma_i (m_1 + \cdots + m_d) - m_i.$$

Thus, $(v_i + v_j) c_j^{(i)} = \gamma_i - \delta_{ij}$ and we have

$$Sm_i = \frac{\gamma_i - 1}{2 v_i} m_i + \sum_{j \ne i} \frac{\gamma_i}{v_i + v_j} m_j.$$

This proves (c).

Next we prove the formula for γ_i given in (b). Since $E = SE$ we find that

$$\gamma_i \sum_{j=1}^d m_j = Em_i = SEm_i = \gamma_i \sum_{j=1}^d \Big\{ \frac{\gamma_j - 1}{2 v_j} + \sum_{k \ne j} \frac{\gamma_k}{v_j + v_k} \Big\} m_j,$$

for $i = 1, \dots, r$. Note that some γ_i is non–zero, since $EM \ne 0$. Thus, comparing the coefficient of m_j on both sides shows that

$$\sum_{k=1}^d \frac{\gamma_k}{v_j + v_k} = 1 + \frac{1}{2 v_j},$$

for $j = 1, \dots, d$.

We claim that $\det \big(\frac{1}{v_i + v_j} \big)_{1 \le i,j \le d} = \Big(\prod_{i=1}^d 2 v_i \Big)^{-1} \prod_{i > j} \big(\frac{v_i - v_j}{v_i + v_j} \big)^2$. To see this, observe that

$$\big(\prod_{i=1}^d 2 v_i \big) \prod_{i > j} (v_i + v_j)^2 \det \big(\frac{1}{v_i + v_j} \big)_{1 \le i,j \le d}$$

is a symmetric polynomial in $v_1, \dots, v_d$ which is divisible by $v_i - v_j$ for $i \ne j$. This shows that this determinant is a constant multiple of $\prod_{i>j} \big(v_i - v_j \big)^2$. To determine the constant, we multiply $\det \big(\frac{1}{v_i + v_j} \big)_{1 \le i,j \le d}$ by v_n, set $v_n = \infty$ and use induction.

By the last paragraph, the matrix $\left(\frac{1}{v_i+v_j}\right)_{1\le i,j\le d}$ is invertible, so $\gamma_1,\dots,\gamma_d$ are uniquely determined. Hence, to prove the formula for γ_i it suffices to show that

$$\sum_{k=1}^{d}\frac{2v_k-(-1)^d}{v_j+v_k}\prod_{i\neq k}\frac{v_k+v_i}{v_k-v_i}=1+\frac{1}{2v_j},$$

for $1\le j\le d$. Let $f(z)=\frac{2z-(-1)^d}{2z(z+v_j)}\prod_{i=1}^d\frac{z+v_i}{z-v_i}$ and view $f(z)$ as an element of the function field of the projective line defined over $F(v_1,\dots,v_d)$. Then, the left hand side can be interpreted as the sum $\sum_{k=1}^d \operatorname{Res}_{z=v_k} f(z)dz$, where $\operatorname{Res}_{z=v} f(z)dz$ is the residue of $f(z)$ at v, if $v\neq\infty$, and it is the residue of $-\frac{1}{z^2}f(\frac{1}{z})$ at 0, if $v=\infty$. Thus, the residue theorem for complete non–singular curves implies that

$$\sum_{k=1}^{d}\frac{2v_k-(-1)^d}{v_j+v_k}\prod_{i\neq k}\frac{v_k+v_i}{v_k-v_i}=-\Big(\operatorname*{Res}_{z=\infty}f(z)dz+\operatorname*{Res}_{z=0}f(z)dz\Big)=1+\frac{1}{2v_j},$$

as required. Hence, we have shown that, for $1\le j\le d$,

$$\gamma_j=(2v_j-(-1)^d)\prod_{k\neq j}\frac{v_j+v_k}{v_j-v_k}\,,$$

so (b) is proved. (For a combinatorial proof see Proposition 4.21(a) below.)

Now, since $Em=\omega_0 m$ and $m=\sum_{i=1} m_i$, we have that $\omega_0=\sum_{i=1}^d\gamma_i$. Similarly, we have that $\omega_a=\sum_{j=1}^m v_j^a\gamma_j$ because

$$\begin{aligned}\omega_a m&=\frac{\omega_a}{\omega_0}Em=\frac{1}{\omega_0}EX_1^aEm=EX_1^am\\&=\sum_{i=1}^d EX_1^am_i=\sum_{i=1}^d v_i^aEm_i=\Big(\sum_{i=1}^d v_i^a\gamma_i\Big)m.\end{aligned}$$

We now show that

$$\omega_0=\sum_{i=1}^d\gamma_i=\begin{cases}2(v_1+\dots+v_d), & \text{if } d \text{ is even,}\\ 2(v_1+\dots+v_d)+1, & \text{if } d \text{ is odd.}\end{cases}$$

To evaluate $\sum_{i=1}^d\gamma_i$, we consider $g(z)=\frac{2z-(-1)^d}{2z}\prod_{i=1}^d\frac{z+v_i}{z-v_i}$ and interpret the sum as $\sum_{i=1}^d\operatorname{Res}_{z=v_i}g(z)dz$. Then the residue theorem gives the desired formula for ω_0.

We next show that M is uniquely determined, up to isomorphism. Suppose that $\mathscr{W}_{r,n}(\mathbf{u})$ has another irreducible module of dimension d' upon which e acts non–trivially. Then, by the argument above,

$$\omega_0=\begin{cases}2(v'_1+\dots+v'_{d'}), & \text{if } d' \text{ is even,}\\ 2(v'_1+\dots+v'_{d'})+1, & \text{if } d' \text{ is odd,}\end{cases}$$

for some $v'_1,\dots,v'_{d'}\subseteq\{u_1,\dots,u_r\}$. As we are assuming that $u_1,\dots,u_r$ are algebraically independent, this forces $d'=d$ and $v'_i=v_{(i)\sigma}$, for some $\sigma\in\mathfrak{S}_d$ and $1\le i\le d$. Hence, by (a)–(c), $M\cong M'$ as required.

Finally, it remains to verify that (a)–(c) define a representation of $\mathscr{W}_{r,2}(\mathbf{u})$ whenever $\omega_a=\sum_{i=1}^d v_i^a\gamma_i$, for $a\ge 0$ and γ_i as above. It is easy to check that the action respects the relations $E(X_1+X_2)=0=(X_1+X_2)E$, $EX_1^aE=\omega_aE$ and $X_1S-SX_2=E-1=SX_1-X_2S$. That $SE=E$ and $ES=E$ on M, follows from the identity $\sum_{k=1}^d\frac{\gamma_k}{v_j+v_k}=1+\frac{1}{2v_j}$ proved above. We now prove that $S^2=1$. Observe that S^2 commutes with X_1 when acting on M. As the v_i are pairwise distinct, we have $S^2m_i=c_im_i$, for some $c_i\in R$. Explicit computation shows that $c_i = \frac{1-2\gamma_i}{4v_i^2} + \gamma_i \sum_{j=1}^d \frac{\gamma_j}{(v_i+v_j)^2}$. Computing the residues of $h(z)dz$, when $h(z) = \frac{2z-(-1)^d}{2z(z+v_i)^2} \prod_{k=1}^d \frac{z+v_i}{z-v_i}$, proves that $c_i = 1$, for $1 \le i \le d$. □

Theorem 3.3. *Let F be a field and suppose that $2 \nmid \operatorname{char} F$ and that $u_1, \dots, u_r$ are algebraically independent over F. Let $R = F(u_1, \dots, u_r)$ and suppose that $\mathscr{W}_{r,2}(\mathbf{u})$ is a split semisimple R–algebra and that $\omega_0 \neq 0$. Then $\mathscr{W}_{r,2}(\mathbf{u})$ has dimension $3r^2 = r^n(2n-1)!!\,|_{n=2}$ if and only if, for all $a \ge 0$,*

$$\omega_a = \sum_{j=1}^r u_j^a \gamma_j,$$

where $\gamma_i = (2u_i - (-1)^r) \prod_{\substack{1 \le j \le r \\ j \ne i}} \frac{u_i + u_j}{u_i - u_j}$.

Proof. We have constructed all the irreducible $\mathscr{W}_{r,2}$-modules in Propositions 3.1 and 3.2 above. Under our assumptions, Proposition 3.1 implies that $\mathscr{W}_{r,2}(\mathbf{u})$ has (a) $2r$ pairwise non–isomorphic one dimensional representations and (b) $\binom{r}{2}$ pairwise non–isomorphic two dimensional representations. Note that case (c) from Proposition 3.1 does not occur since $u_1, \dots, u_r$ are pairwise distinct. Further, Proposition 3.2 implies that $\mathscr{W}_{r,2}(\mathbf{u})$ has a unique irreducible representation M such that $EM \neq 0$ and, moreover, if $d = \dim M$ then $1 \le d \le r$. Hence, by the Wedderburn–Artin theorem we have

$$\dim \mathscr{W}_{r,2}(\mathbf{u}) = 2r + 4\tbinom{r}{2} + d^2 = 2r^2 + d^2,$$

where $d = r$ if and only if ω_a, for $a \ge 0$, is given by the formulae in the statement of the Theorem. Hence, $\dim \mathscr{W}_{r,2}(\mathbf{u}) = 3r^2$ if and only if $d = r$. The result follows. □

Recall that Schur's q–functions $q_a = q_a(\mathbf{x})$ in the indeterminates $\mathbf{x} = (x_1, \dots, x_r)$ [Mac95, p. 250] are defined by the equation

$$\prod_{i=1}^r \frac{1 + x_i y}{1 - x_i y} = \sum_{a \ge 0} q_a(\mathbf{x}) y^a.$$

Note that $q_a(\mathbf{x})$ is a polynomial in $\mathbf{x}$, for all $a \ge 0$.

Lemma 3.4. *Assume that R is an integral domain and that 2 is invertible in R. Suppose that $\mathbf{u} \in R^r$, with $u_i - u_j \neq 0$ whenever $i \neq j$. Let F be the quotient field of R and for $a \ge 0$ define*

$$\omega_a = \sum_{i=1}^r \big(2u_i - (-1)^r\big) u_i^a \prod_{\substack{1 \le j \le r \\ j \ne i}} \frac{u_i + u_j}{u_i - u_j} \in F,$$

as in Theorem 3.3. Then $\omega_a = q_{a+1}(\mathbf{u}) - \frac{1}{2}(-1)^r q_a(\mathbf{u}) + \frac{1}{2}\delta_{a0}$. In particular, $\omega_a \in R$.

Proof. If $a = 0$ then the result follows from Proposition 3.2, so we can assume that $a > 0$. Let $f(z) = \frac{1}{2} z^{a-1}(2z - (-1)^r) \prod_{i=1}^r \frac{z+u_i}{z-u_i}$. Then ω_a can be interpreted as $\sum_{i=1}^r \operatorname{Res}_{z=u_i} f(z)\,dz = -\operatorname{Res}_{z=\infty} f(z)\,dz$. Calculating the residue of $f(z)dz$ at $z = \infty$ now shows that $\omega_a = q_{a+1}(\mathbf{u}) - \frac{1}{2}(-1)^r q_a(\mathbf{u}) + \frac{1}{2}\delta_{a0}$. (See [Mac95, (2.9), p. 209] for a more direct proof.) Hence, $\omega_a \in R$ since $q_b(\mathbf{x}) \in R[\mathbf{x}]$, for $b \ge 0$. □

We want the cyclotomic Wenzl algebras to be "cyclotomic" generalizations of the Brauer algebras, so we want them to be free of rank $r^n(2n-1)!!$. Theorem 3.3 gives sufficient conditions on $\Omega = \{\, \omega_a \mid a \ge 0 \,\}$ for $\mathscr{W}_{r,2}(\mathbf{u})$ to have dimension $r^n(2n-1)!!$ when R is an algebraically closed field and $n = 2$. It seems reasonable therefore to require in our study of $\mathscr{W}_{r,n}(\mathbf{u})$ that Ω satisfy the following property.

Definition 3.5. Let $\Omega = \{ \omega_a \mid a \ge 0 \} \subseteq R$ and suppose that $\mathbf{u} \in R^r$. Then Ω is $\mathbf{u}$–admissible if $\omega_a = q_{a+1}(\mathbf{u}) - \frac{1}{2}(-1)^r q_a(\mathbf{u}) + \frac{1}{2}\delta_{a0}$, for $a \ge 0$.

Lemma 3.6. *Suppose that $\mathbf{u} \in R^r$ and that Ω is $\mathbf{u}$–admissible. Let y be an indeterminate and define $\widetilde{W}_1(y) = \sum_{a\ge 0} \omega_a y^{-a}$. Then*

$$\widetilde{W}_1(y) + y - \frac{1}{2} = (y - \frac{1}{2}(-1)^r) \prod_{i=1}^{r} \frac{y+u_i}{y-u_i}.$$

Proof. By definition, $\widetilde{W}_1(y) = \frac{1}{2} + \sum_{a\ge 0} \big(q_{a+1}(\mathbf{u}) - \frac{1}{2}(-1)^r q_a(\mathbf{u})\big) y^{-a}$. Now expand this equation using the definition of the Schur q–functions. □

Corollary 3.7. *Suppose that Ω is $\mathbf{u}$–admissible. Then Ω is admissible.*

Proof. First suppose that $\mathbf{x} = (x_1, \dots, x_r)$ are algebraically independent and let $\Omega = \{ \omega_a \mid a \ge 0 \}$, where $\omega_a = q_{a+1}(\mathbf{x}) - \frac{1}{2}(-1)^r q_a(\mathbf{x}) + \frac{1}{2}\delta_{a0}$, for $a \ge 0$. Then Ω is $\mathbf{x}$–admissible by definition and hence admissible by Corollary 2.4 and Proposition 3.2. Therefore, by the definition of admissibility we have the following polynomial identity in $x_1, \dots, x_r$

$$\omega_{2a+1} = \frac{1}{2}\Big\{ -\omega_{2a} + \sum_{b=1}^{2a+1} (-1)^{b-1} \omega_{b-1}\omega_{2a+1-b} \Big\}.$$

The general case now follows by specializing $x_i = u_i$, for $1 \le i \le r$.

For a second proof, note that if Ω is $\mathbf{u}$–admissible then

$$\big(\widetilde{W}_1(y) + y - \frac{1}{2}\big)\big(\widetilde{W}_1(-y) - y - \frac{1}{2}\big) = (\frac{1}{2} - y)(\frac{1}{2} + y),$$

by Lemma 3.6. Hence, Ω is admissible by Remark 2.11. □

4. The seminormal representations of $\mathscr{W}_{r,n}(\mathbf{u})$

In this section, we will give an explicit description of the irreducible representations of $\mathscr{W}_{r,n}(\mathbf{u})$ in the special case when R is an field of characteristic greater than $2n$ and when the parameters $\mathbf{u}$ satisfy some rather technical assumptions; see Theorem 4.13.

The semisimple irreducible representations of the Brauer algebra $\mathscr{B}_n(\omega)$ are labelled by partitions of $n-2m$, where $0 \le m \le \lfloor \frac{n}{2} \rfloor$, and a basis of the representation indexed by the partition λ is indexed by the set of updown λ–tableaux. Analogously, we might expect that the semisimple irreducible representations of $\mathscr{W}_{r,n}(\mathbf{u})$ should be indexed by the multipartitions of $n-2m$, with the bases of these modules being indexed by the updown λ–tableaux, where λ is a multipartition. We will see that this is the case. We begin by defining these combinatorial objects.

Recall that a partition of m is a sequence of weakly decreasing sequence of non–negative integers $\tau = (\tau_1, \tau_2, \dots)$ such that $|\tau| := \tau_1 + \tau_2 + \cdots = m$. Similarly, an r–multipartition of m, or more simply a multipartition, is an ordered r–tuple $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of partitions $\lambda^{(s)}$, with $|\lambda| := |\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = m$. If λ is a multipartition of m then we write $\lambda \vdash m$.

If λ and μ are two multipartitions we say that μ is obtained from λ by adding a box if there exists a pair (i,s) such that $\mu_i^{(s)} = \lambda_i^{(s)} + 1$ and $\mu_j^{(t)} = \lambda_j^{(t)}$ for $(j,t) \ne (i,s)$. In this situation we will also say that λ is obtained from μ by removing a box and we write $\lambda \subset \mu$ and $\mu \setminus \lambda = (i, \lambda_i^{(s)}, s)$. We will also say that the triple $(i, \lambda_i^{(s)}, s)$ is an addable node of λ and a removable node of μ. Note that $|\mu| = |\lambda| + 1$.

Fix an integer m with $0 \le m \le \lfloor \frac{n}{2} \rfloor$ and let λ be a multipartition of $n-2m$. An n–updown λ–tableau, or more simply an updown λ–tableau, is a sequence $\mathfrak{u} =$

$(\mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_n)$ of multipartitions where $\mathfrak{u}_n = \lambda$ and the multipartition $\mathfrak{u}_i$ is obtained from $\mathfrak{u}_{i-1}$ by either *adding* or *removing* a box, for $i = 1, \dots, n$. For convenience we set $\mathfrak{u}_0$ equal to the empty multipartition $\emptyset$. Let $\mathscr{T}_n^{ud}(\lambda)$ be the set of updown λ–tableaux of n. Note that λ is a multipartition of $n - 2m$ and each element of $\mathscr{T}_n^{ud}(\lambda)$ is an n–tuple of multipartitions, so the n is necessary in this notation.

In the special case when λ is a multipartition of n (so $m = 0$), we can identify updown λ–tableaux with standard λ–tableaux. This is the origin of this terminology.

Definition 4.1. Suppose $1 \le k \le n$. Define an equivalence relation $\overset{k}{\sim}$ on $\mathscr{T}_n^{ud}(\lambda)$ by declaring that $\mathfrak{u} \overset{k}{\sim} \mathfrak{t}$ if $\mathfrak{u}_j = \mathfrak{t}_j$ whenever $1 \le j \le n$ and $j \ne k$, for $\mathfrak{t}, \mathfrak{u} \in \mathscr{T}_n^{ud}(\lambda)$.

The following result is an immediate consequence of Definition 4.1.

Lemma 4.2. *Suppose $s \in \mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$. Then there is a bijection between the set of all addable and removable nodes of $\mathfrak{t}_{k-1}$ and the set of $\mathfrak{u} \in \mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{u} \overset{k}{\sim} \mathfrak{t}$.*

Let λ be a multipartition and suppose that $\mathfrak{u}$ is an n–updown λ–tableaux. For $k = 2, \dots, n$ the mutipartitions $\mathfrak{u}_k$ and $\mathfrak{u}_{k-1}$ differ by exactly one box; so either $\mathfrak{u}_k \subset \mathfrak{u}_{k-1}$ or $\mathfrak{u}_{k-1} \subset \mathfrak{u}_k$. We define the $\mathsf{content}$ of k in $\mathfrak{u}$ to be element $c_\mathfrak{u}(k) \in R$ given by

$$c_\mathfrak{u}(k) = \begin{cases} j - i + u_s, & \text{if } \mathfrak{u}_k \backslash \mathfrak{u}_{k-1} = (i,j,s), \\ i - j - u_s, & \text{if } \mathfrak{u}_{k-1} \backslash \mathfrak{u}_k = (i,j,s). \end{cases}$$

More generally, if $\alpha = (i,j,s)$ is an addable node of λ we define $c(\alpha) = u_s + j - i$ and if α is a removable node of λ we set $c(\alpha) = -(u_s + j - i)$.

The key property of contents that we need to construct the seminormal representations is the following. Note that we are not (yet) assuming that R is a field.

Definition 4.3. The parameters $\mathbf{u} = (u_1, \dots, u_r)$ are **generic** for $\mathscr{W}_{r,n}(\mathbf{u})$ if whenever there exists $d \in \mathbb{Z}$ such that either $u_i \pm u_j = d \cdot 1_R$ and $i \ne j$, or $2u_i = d \cdot 1_R$ then $|d| \ge 2n$.

For example, $\mathbf{u}$ is generic for $\mathscr{W}_{r,n}(\mathbf{u})$ if $u_1, \dots, u_r$ are algebraically independent over a subfield of R.

Lemma 4.4. *Suppose that the parameters $\mathbf{u}$ are generic for $\mathscr{W}_{r,n}(\mathbf{u})$ and that $\operatorname{char} R \ge 2n$. Let λ be a multipartition of $n - 2m$, where $0 \le m \le \lfloor \frac{n}{2} \rfloor$, and suppose that $\mathfrak{t}, \mathfrak{u} \in \mathscr{T}_n^{ud}(\lambda)$. Then*

a) $\mathfrak{t} = \mathfrak{u}$ if and only if $c_\mathfrak{t}(k) = c_\mathfrak{u}(k)$, for $k = 1, \dots, n$;

b) if $1 \le k < n$ then $c_\mathfrak{t}(k) - c_\mathfrak{t}(k+1) \ne 0$; and,

c) if $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$ then $c_\mathfrak{t}(k) \pm c_\mathfrak{u}(k) \ne 0$, whenever $\mathfrak{u} \overset{k}{\sim} \mathfrak{t}$ and $\mathfrak{u} \ne \mathfrak{t}$.

Proof. Part (a) follows by induction on n. Parts (b) and (c) are immediate from the definitions. The key point is that our assumptions imply that the contents of the nodes in λ are distinct and they determine the order in which nodes are added or removed. ∎

Until further notice we fix an integer m with $0 \le m \le \lfloor \frac{n}{2} \rfloor$ and we fix a multipartition λ of $n - 2m$.

Motivated by [Naz96], we introduce the following rational functions in an indeterminate y. These functions will play a key role in the construction of seminormal representations of $\mathscr{W}_{r,n}(\mathbf{u})$.

Definition 4.5. Suppose that $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$. For $1 \le k \le n$, define rational functions $W_k(y, \mathfrak{t})$ by

$$W_k(y,\mathfrak{t}) = \frac{1}{2} - y + \big(y - \frac{1}{2}(-1)^r\big) \prod_\alpha \frac{y + c(\alpha)}{y - c(\alpha)},$$

where α runs over the addable and removable nodes of the multipartition $\mathfrak{t}_{k-1}$.

The rational functions $W_k(y, \mathfrak{t})$ are related to the combinatorics above by the following result. If $f(y)$ is a rational function and $\alpha \in R$ then we write $\operatorname{Res}_{y=\alpha} f(y)$ for the residue of $f(y)$ at $y = \alpha$.

Lemma 4.6. *Suppose that* $\mathbf{u}$ *is generic and* $\operatorname{char} R \ge 2n$. *Let* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ *and* $1 \le k \le n$. *Then*

$$\frac{W_k(y,\mathfrak{t})}{y} = \sum_\alpha \Big(\operatorname*{Res}_{y=c(\alpha)} \frac{W_k(y,\mathfrak{t})}{y}\Big) \cdot \frac{1}{y - c(\alpha)},$$

where α *runs over the addable and removable nodes of* $\mathfrak{t}_{k-1}$.

Proof. As the $c(\alpha)$ are pairwise distinct, we can certainly write

$$\frac{W_k(y,\mathfrak{t})}{y} = a + \frac{b}{y} + \sum_\alpha \Big(\operatorname*{Res}_{y=c(\alpha)} \frac{W_k(y,\mathfrak{t})}{y}\Big) \cdot \frac{1}{y - c(\alpha)},$$

for some $a, b \in R$, where α runs over the addable and removable nodes of $\mathfrak{t}_{k-1}$. Now, $a = \frac{W_k(y,\mathfrak{t})}{y} |_{y=\infty} = 0$. Let c be the number of addable and removable nodes of $\mathfrak{t}_{k-1}$. Since a partition always has an odd number of addable and removable nodes, we have that $(-1)^c = (-1)^r$. Therefore,

$$b = \operatorname*{Res}_{y=0} \frac{W_k(y,\mathfrak{t})}{y} = \frac{1}{2}\big(1 - (-1)^c(-1)^r\big) = 0,$$

as we needed to show. □

We are now ready to define the matrices which make up the seminormal form.

Definition 4.7. Let λ be a multipartition and k an integer with $1 \le k \le n$. Suppose that $\mathfrak{t}$ and $\mathfrak{u}$ are updown λ–tableaux in $\mathscr{T}_n^{ud}(\lambda)$ such that $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$. Then we define the scalars $e_{\mathfrak{t}\mathfrak{u}}(k) \in R$ by

$$e_{\mathfrak{t}\mathfrak{u}}(k) = \begin{cases} \operatorname*{Res}_{y=c_\mathfrak{t}(k)} \dfrac{W_k(y,\mathfrak{t})}{y}, & \text{if } \mathfrak{t} = \mathfrak{u}, \\ \sqrt{e_{\mathfrak{t}\mathfrak{t}}(k)}\sqrt{e_{\mathfrak{u}\mathfrak{u}}(k)}, & \text{if } \mathfrak{t} \ne \mathfrak{u} \text{ and } \mathfrak{u} \overset{k}{\sim} \mathfrak{t}, \\ 0, & \text{otherwise.} \end{cases}$$

(In (4.12) below we will fix the choice of square roots $\sqrt{e_{\mathfrak{t}\mathfrak{t}}(k)}$, for $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ and $1 \le k \le n$.)

We remark that if $\mathfrak{t}_{k-1} \ne \mathfrak{t}_{k+1}$ then the definition of $e_{\mathfrak{t}\mathfrak{u}}(k)$ still makes sense, however, we do not define $e_{\mathfrak{t}\mathfrak{u}}(k)$ in this case as we will not need it (see Theorem 4.13 below).

It follows from Definition 4.5 that

$$e_{\mathfrak{t}\mathfrak{t}}(k) = \big(2c_\mathfrak{t}(k) - (-1)^r\big) \prod_\alpha \frac{c_\mathfrak{t}(k) + c(\alpha)}{c_\mathfrak{t}(k) - c(\alpha)} \tag{4.8}$$

where α runs over all addable and removable nodes of $\mathfrak{t}_{k-1}$ with $c(\alpha) \ne c_\mathfrak{t}(k)$. Note that Lemma 4.4 now implies that if $\mathfrak{u} \overset{k}{\sim} \mathfrak{t}$ then $e_{\mathfrak{t}\mathfrak{u}}(k) \ne 0$, for $1 \le k < n$. This will be used many times below. We also observe that Lemma 4.6 can be restated as

$$\frac{W_k(y,\mathfrak{t})}{y} = \sum_{\mathfrak{u}\overset{k}{\sim}\mathfrak{t}} \frac{e_{\mathfrak{u}\mathfrak{u}}(k)}{y-c_{\mathfrak{u}}(k)}. \tag{4.9}$$

Given two partitions $\mathfrak{t}$ and $\mathfrak{u}$ write $\mathfrak{t}\ominus\mathfrak{u}=\alpha$ if either $\mathfrak{u}\subset\mathfrak{t}$ and $\mathfrak{t}\setminus\mathfrak{u}=\alpha$, or $\mathfrak{t}\subset\mathfrak{u}$ and $\mathfrak{u}\setminus\mathfrak{t}=\alpha$.

Definition 4.10. Let $\mathfrak{t}\in\mathscr{T}^{ud}_n(\lambda)$ and suppose that $\mathfrak{t}_{k-1}\neq\mathfrak{t}_{k+1}$, for some k with $1\le k<n$.

a) We define

$$a_{\mathfrak{t}}(k)=\frac{1}{c_{\mathfrak{t}}(k+1)-c_{\mathfrak{t}}(k)} \qquad\text{and}\qquad b_{\mathfrak{t}}(k)=\sqrt{1-a_{\mathfrak{t}}(k)^2}.$$

(We fix the choice of square root for $b_{\mathfrak{t}}(k)$ in (4.12) below. Note that $c_{\mathfrak{t}}(k+1)-c_{\mathfrak{t}}(k)\neq 0$ by Lemma 4.4(b).)

b) If $\mathfrak{t}_k\ominus\mathfrak{t}_{k-1}$ and $\mathfrak{t}_{k+1}\ominus\mathfrak{t}_k$ are in different rows and in different columns then we define $S_k\mathfrak{t}$ to be the updown λ–tableau

$$S_k\mathfrak{t}=(\mathfrak{t}_1,\cdots,\mathfrak{t}_{k-1},\mathfrak{u}_k,\mathfrak{t}_{k+1},\cdots,\mathfrak{t}_n)$$

where $\mathfrak{u}_k$ is the multipartition which is uniquely determined by the conditions $\mathfrak{u}_k\ominus\mathfrak{t}_{k+1}=\mathfrak{t}_{k-1}\ominus\mathfrak{t}_k$ and $\mathfrak{t}_{k-1}\ominus\mathfrak{u}_k=\mathfrak{t}_k\ominus\mathfrak{t}_{k+1}$. If the nodes $\mathfrak{t}_k\ominus\mathfrak{t}_{k-1}$ and $\mathfrak{t}_{k+1}\ominus\mathfrak{t}_k$ are both in the same row, or both in the same column, then $S_k\mathfrak{t}$ is not defined.

We remark that if $\mathfrak{t}_{k-1}=\mathfrak{t}_{k+1}$ then the definitions of $a_{\mathfrak{t}}(k)$ and $b_{\mathfrak{t}}(k)$ both make sense, however, we do not define them in this case as we will never need them (see Theorem 4.13 below). Moreover, the condition $\mathfrak{t}_{k-1}\neq\mathfrak{t}_{k+1}$ is crucial in proving Lemma 4.11(b) below. (In fact, if we drop this condition then Lemma 4.11(b) is not correct.)

We leave the following Lemma as an exercise to help the reader familiarize themselves with the definitions.

Lemma 4.11. *Suppose that $\mathfrak{t}\in\mathscr{T}^{ud}_n(\lambda)$ and $1\le k<n$. Then:*

a) *If $S_k\mathfrak{t}$ is defined then $c_{\mathfrak{t}}(k)=c_{S_k\mathfrak{t}}(k+1)$ and $c_{\mathfrak{t}}(k+1)=c_{S_k\mathfrak{t}}(k)$; consequently, $a_{S_k\mathfrak{t}}(k)=-a_{\mathfrak{t}}(k)$.*

b) *If $S_k\mathfrak{t}$ is not defined then $a_{\mathfrak{t}}(k)=\pm1$ and $b_{\mathfrak{t}}(k)=0$.*

Finally, if $\mathfrak{t}_{k-1}=\mathfrak{t}_{k+1}$ and $\mathfrak{u}\overset{k}{\sim}\mathfrak{t}$, where $1\le k<n$, we set

$$s_{\mathfrak{t}\mathfrak{u}}(k)=\frac{e_{\mathfrak{t}\mathfrak{u}}(k)-\delta_{\mathfrak{t}\mathfrak{u}}}{c_{\mathfrak{t}}(k)+c_{\mathfrak{u}}(k)}.$$

Note that $c_{\mathfrak{t}}(k)+c_{\mathfrak{u}}(k)\neq0$ by Lemma 4.4.

We will assume that we have chosen the square roots in the definitions of $b_{\mathfrak{t}}(k)$ and $e_{\mathfrak{t}\mathfrak{u}}(k)$ so that the following equalities hold.

Assumption 4.12 (Root conditions)**.** *We assume that the ring R is large enough so that $\sqrt{e_{\mathfrak{t}\mathfrak{t}}(k)}\in R$ and $b_{\mathfrak{t}}(k)=\sqrt{1-a_{\mathfrak{t}}(k)^2}\in R$, for all $\mathfrak{t},\mathfrak{u}\in\mathscr{T}^{ud}_n(\lambda)$ and $1\le k<n$, and that the following equalities hold:*

a) *If $\mathfrak{t}_{k-1}\neq\mathfrak{t}_{k+1}$ and $S_k\mathfrak{t}$ is defined then $b_{S_k\mathfrak{t}}(k)=b_{\mathfrak{t}}(k)$.*

b) *If $\mathfrak{t}_{k-1}\neq\mathfrak{t}_{k+1}$ and $\mathfrak{t}\overset{l}{\sim}\mathfrak{u}$, where $|k-l|>1$, then $b_{\mathfrak{t}}(k)=b_{\mathfrak{u}}(k)$.*

c) *If $\mathfrak{t}_{k-1}\neq\mathfrak{t}_{k+1}$, $\mathfrak{t}_k\neq\mathfrak{t}_{k+2}$ and $S_k\mathfrak{t}$ and $S_{k+1}\mathfrak{t}$ are both defined then $b_{S_{k+1}\mathfrak{t}}(k)=b_{S_k\mathfrak{t}}(k+1)$.*

d) *If $\mathfrak{t}_{k-1}=\mathfrak{t}_{k+1}$ and $\mathfrak{t}_k=\mathfrak{t}_{k+2}$ then $\sqrt{e_{\mathfrak{t}\mathfrak{t}}(k)}\sqrt{e_{\mathfrak{t}\mathfrak{t}}(k+1)}=1$.*

e) *If $\mathfrak{t}_{k-1}=\mathfrak{t}_{k+1}$, $\mathfrak{u}_{k-1}=\mathfrak{u}_{k+1}$ and $e_{\mathfrak{t}\mathfrak{t}}(k)=e_{\mathfrak{u}\mathfrak{u}}(k)$ then $\sqrt{e_{\mathfrak{t}\mathfrak{t}}(k)}=\sqrt{e_{\mathfrak{u}\mathfrak{u}}(k)}$.*

f) If $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$, $\mathfrak{t}_k = \mathfrak{t}_{k+2}$ *and* $\mathfrak{u} \overset{k+1}{\sim} \mathfrak{t}$, $\mathfrak{w} \overset{k}{\sim} \mathfrak{t}$ *with* $S_k\mathfrak{u}$ *and* $S_{k+1}\mathfrak{w}$ *both defined and* $S_k\mathfrak{u} = S_{k+1}\mathfrak{w}$ *then* $b_{\mathfrak{u}}(k)\sqrt{e_{\mathfrak{u}\mathfrak{u}}(k+1)} = b_{\mathfrak{w}}(k+1)\sqrt{e_{\mathfrak{w}\mathfrak{w}}(k)}$.

In Lemma 5.3 below we show that if $R = \mathbb{R}$ then it is possible to choose $\mathbf{u}$ so that the Root Condition is satisfied.

Assuming (4.12) we can now give the formulas for the seminormal representations of $\mathscr{W}_{r,n}(\mathbf{u})$.

Theorem 4.13. *Suppose that* R *is a field such that* $\operatorname{char} R \geq 2n$ *and that the root conditions (4.12) hold in* R*. Assume that* $\mathbf{u}$ *is generic for* $\mathscr{W}_{r,n}(\mathbf{u})$*. Let* $\Delta(\lambda)$ *be the* R*–vector space with basis* $\{ v_{\mathfrak{t}} \mid \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda) \}$*. Then* $\Delta(\lambda)$ *becomes a* $\mathscr{W}_{r,n}(\mathbf{u})$*–module via*

- $$S_k v_{\mathfrak{t}} = \begin{cases} \displaystyle\sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} s_{\mathfrak{t}\mathfrak{u}}(k) v_{\mathfrak{u}}, & \text{if } \mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}, \\ a_{\mathfrak{t}}(k) v_{\mathfrak{t}} + b_{\mathfrak{t}}(k) v_{S_k\mathfrak{t}}, & \text{if } \mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}, \end{cases}$$
- $$E_k v_{\mathfrak{t}} = \begin{cases} \displaystyle\sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} e_{\mathfrak{t}\mathfrak{u}}(k) v_{\mathfrak{u}}, & \text{if } \mathfrak{t}_{k-1} = \mathfrak{t}_{k+1} \\ 0, & \text{if } \mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}, \end{cases}$$
- $X_j v_{\mathfrak{t}} = c_{\mathfrak{t}}(j) v_{\mathfrak{t}}$,

for $1 \leq k < n$ *and* $1 \leq j \leq n$ *and where we set* $v_{S_k\mathfrak{t}} = 0$ *if* $S_k\mathfrak{t}$ *is not defined.*

Definition 4.14. We call $\Delta(\lambda)$ a seminormal representation of $\mathscr{W}_{r,n}(\mathbf{u})$.

We note that the action of the operators E_k and S_k on $\Delta(\lambda)$, with respect to the basis $\{ v_{\mathfrak{t}} \mid \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda) \}$, are given by symmetric matrices, for $0 \leq k < n$.

For the remainder of this section we assume that R is an algebraically closed field with $\operatorname{char} R \geq 2n$ and that the parameters $\mathbf{u}$ are generic for $\mathscr{W}_{r,n}(\mathbf{u})$ and satisfy (4.12). The proof of Theorem 4.13 will occupy the rest of this section. Our strategy is to use the rational functions $W_k(\mathfrak{t}, k)$ to verify that the action that we have just defined of $\mathscr{W}_{r,n}(\mathbf{u})$ on $\Delta(\lambda)$ respects all of the relations in $\mathscr{W}_{r,n}(\mathbf{u})$.

Throughout this section it will be convenient to work with formal (infinite) linear combinations of elements of $\Delta(\lambda)$ and $\mathscr{W}_{r,n}(\mathbf{u})$; alternatively, the reader may prefer to think that we have extended our coefficient ring from R to $R((y^{-1}))$, where y is an indeterminate over R. In fact, at times we will need to work with formal series involving more than one indeterminate.

If A is an algebra we let $Z(A)$ be its center.

Lemma 4.15. *Suppose* $k \geq 0$ *and that* $a \geq 0$*. Then there exist elements* $\omega_k^{(a)}$ *in* $Z\big(\mathscr{W}_{r,k-1}(\mathbf{u})\big) \cap R[X_1, \dots, X_{k-1}]$ *such that*

$$E_k X_k^a E_k = \omega_k^{(a)} E_k.$$

Moreover, the generating series $\widetilde{W}_k(y) = \sum_{a \geq 0} \omega_k^{(a)} y^{-a}$ *satisfies*

$$\widetilde{W}_{k+1}(y) = -y + \frac{1}{2} + \frac{(y+X_k)^2 - 1}{(y-X_k)^2 - 1} \frac{(y-X_k)^2}{(y+X_k)^2} \big(\widetilde{W}_k(y) + y - \frac{1}{2}\big).$$

Proof. Observe that $\sum_{a \geq 0} E_k X_k^a E_k y^{-a} = E_k \frac{y}{y - X_k} E_k$, so to prove the Lemma it is enough to argue by induction on k to show that $E_k \frac{1}{y - X_k} E_k = \frac{1}{y} \widetilde{W}_k(y) E_k$, where $\widetilde{W}_k(y)$ and its coefficients are as above.

If $k = 1$ then there is nothing to prove. Assume than that $k > 1$. Starting with the identity

$$S_k \frac{1}{y - X_k} = \frac{1}{y - X_{k+1}} S_k + \frac{1}{y + X_k} E_k \frac{1}{y - X_k} - \frac{1}{(y - X_k)(y - X_{k+1})}$$

Nazarov [Naz96, Prop. 4.2] proves that $E_{k+1}\frac{1}{y-X_{k+1}}E_{k+1} = \frac{1}{y}\widetilde{W}_{k+1}(y)E_{k+1}$, where $\widetilde{W}_{k+1}(y)$ satisfies the recurrence relation above. Nazarov assumes that he is working over the complex field (so, $R = \mathbb{C}$), however, his arguments are valid over an arbitrary ring. Nazarov also proves that if $R = \mathbb{C}$ then the coefficients of $\widetilde{W}_k(y)$ are central in $\mathscr{W}_{r,k-1}(\mathbf{u})$. We modify Nazarov's arguments to establish centrality for fields of positive characteristic.

By induction we may assume that the coefficients of $\widetilde{W}_k(y)$ commute with $E_1, \dots, E_{k-2}$ and $S_1, \dots, S_{k-2}$, so it is enough to show that the coefficients of $\widetilde{W}_{k+1}(y)$ commute with E_{k-1} and S_{k-1}. Since $k \geq 2$ we can write

$$\frac{\widetilde{W}_{k+1}(y) + y - \frac{1}{2}}{\widetilde{W}_{k-1}(y) + y - \frac{1}{2}} = \frac{\mathcal{X}}{\mathcal{Y}} := \frac{(y+X_k)^2-1}{(y-X_k)^2-1}\frac{(y-X_k)^2}{(y+X_k)^2}\frac{(y+X_{k-1})^2-1}{(y-X_{k-1})^2-1}\frac{(y-X_{k-1})^2}{(y+X_{k-1})^2}$$

As E_{k-1} and S_{k-1} commute with $\mathscr{W}_{r,k-2}$ it is enough to show that $E_{k-1}\frac{\mathcal{X}}{\mathcal{Y}} = \frac{\mathcal{X}}{\mathcal{Y}}E_{k-1}$ and $S_{k-1}\frac{\mathcal{X}}{\mathcal{Y}} = \frac{\mathcal{X}}{\mathcal{Y}}S_{k-1}$. Now, $E_{k-1}\frac{\mathcal{X}}{\mathcal{Y}} = \frac{\mathcal{X}}{\mathcal{Y}}E_{k-1}$ if and only if $\mathcal{Y}E_{k-1}\mathcal{X} = \mathcal{X}E_{k-1}\mathcal{Y}$, and this follows easily using relation 2.1(i).

To prove that S_{k-1} commutes with $\frac{\mathcal{X}}{\mathcal{Y}}$ let

$$\sum_{m\geq 0} a_m z^m = \frac{(1+X_{k-1}z)(1+X_kz)}{(1-X_{k-1}z)(1-X_kz)},$$

where $z = -y^{-1}$ or $z = (y \pm 1)^{-1}$. Then $a_0 = 1$, $a_1 = 2(X_{k-1}+X_k)$, $a_2 = 2(X_{k-1}+X_k)^2$ and

$$a_m = (X_{k-1}+X_k)a_{m-1} - X_{k-1}X_k a_{m-2}, \quad \text{for } m \geq 3.$$

Consequently, if $m \geq 1$ then $a_m = (X_{k-1}+X_k)f_m(X_{k-1}, X_k)$, for some $f_m \in R[X_{k-1}, X_k]$. Now, relation 2.1(e) implies that S_{k-1} and $X_{k-1}+X_k$ commute. Therefore, by induction,

$$\begin{aligned} S_{k-1}a_m &= (X_{k-1}+X_k)S_{k-1}a_{m-1} - (X_{k-1}X_kS_{k-1} + E_{k-1}X_k - X_kE_{k-1})a_{m-2} \\ &= (X_{k-1}+X_k)a_{m-1}S_{k-1} - X_{k-1}X_ka_{m-2}S_{k-1} \\ &= a_mS_{k-1} \end{aligned}$$

as required. □

Remark 4.16. *To prove that the* $\omega_k^{(a)} \in Z\big(\mathscr{W}_{r,k-1}(\mathbf{u})\big)$ *Nazarov uses the identity*

$$\exp\left(\sum_{a\geq 0} 2\big(X_{k-1}^{2a+1} + X_k^{2a+1}\big)\frac{y^{-2a-1}}{2a+1}\right) = \frac{(y+X_{k-1})(y+X_k)}{(y-X_{k-1})(y-X_k)}.$$

However, this formula is only valid in characteristic zero.

By Lemma 4.15, we have

$$\widetilde{W}_k(y) + y - \frac{1}{2} = \Big(\widetilde{W}_1(y) + y - \frac{1}{2}\Big)\prod_{i=1}^{k-1}\frac{(y+X_i)^2-1}{(y-X_i)^2-1}\cdot\frac{(y-X_i)^2}{(y+X_i)^2}.$$

As the right hand side acts on $\Delta(\lambda)$ as multiplication by a scalar we can define $\widetilde{W}_k(y,\mathfrak{t}) \in R((y^{-1}))$ by $\widetilde{W}_k(y)v_\mathfrak{t} = \widetilde{W}_k(y,\mathfrak{t})v_\mathfrak{t}$.

The next Proposition gives a representation theoretic interpretation of the rational functions $W_k(y,\mathfrak{t})$ which were introduced in Definition 4.5.

Proposition 4.17. *Suppose that* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ *and that* $1 \leq k \leq n$. *Then*

$$W_k(y,\mathfrak{t}) = \widetilde{W}_k(y,\mathfrak{t}).$$

Proof. As Ω is $\mathbf{u}$-admissible, by Lemma 3.6 we have

$$\widetilde{W}_1(y,\mathfrak{t}) + y - \frac{1}{2} = \big(y + \frac{1}{2}(-1)^{r+1}\big) \prod_{t=1}^{r} \frac{y+u_t}{y-u_t}.$$

Consequently, we can rewrite the definition of $\widetilde{W}_k(y,\mathfrak{t})$ as

$$\widetilde{W}_k(y,\mathfrak{t}) + y - \frac{1}{2} = (y - \frac{1}{2}(-1)^r) \cdot \prod_{t=1}^{r} \frac{(y+u_t)}{(y-u_t)} \prod_{i=1}^{k-1} \frac{(y+c_\mathfrak{t}(i))^2-1}{(y-c_\mathfrak{t}(i))^2-1} \cdot \frac{(y-c_\mathfrak{t}(i))^2}{(y+c_\mathfrak{t}(i))^2}.$$

If $c_\mathfrak{t}(i) = -c_\mathfrak{t}(j)$, for some $1 \le i, j \le k-1$ with $i \ne j$, then

$$\frac{(y+c_\mathfrak{t}(i))^2-1}{(y-c_\mathfrak{t}(i))^2-1} \cdot \frac{(y-c_\mathfrak{t}(i))^2}{(y+c_\mathfrak{t}(i))^2} \cdot \frac{(y+c_\mathfrak{t}(j))^2-1}{(y-c_\mathfrak{t}(j))^2-1} \cdot \frac{(y-c_\mathfrak{t}(j))^2}{(y+c_\mathfrak{t}(j))^2} = 1.$$

Hence, in computing $\widetilde{W}_k(y,\mathfrak{t})$ we can assume that $\mathfrak{t} = (\mathfrak{t}_1, \dots, \mathfrak{t}_m, \dots, \mathfrak{t}_k \dots, \mathfrak{t}_n)$ where $m = |s_{k-1}|$, $\mathfrak{t}_m = \mathfrak{t}_{k-1}$ and $c_\mathfrak{t}(i) + c_\mathfrak{t}(i+1) = 0$ for $m < i < k-1$ with $i - m$ odd (so $\mathfrak{t}_{i+1}$ is obtained by adding a box to $\mathfrak{t}_i$, for $1 \le i < m$, and $\mathfrak{t}_i = s_{k-1}$ for $m \le i \le k-1$ with $i-m$ even). Let $\mathfrak{t}_{k-1} = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(r)})$. Fix t with $1 \le t \le r$ and, abusing notation, write $\beta \in \mu_k^{(t)}$ to indicate that $\beta = (i,j,t)$ is a node in row j of $\mu^{(t)}$. Let $p_1 = (k,1,t)$, $p_2 = (k, \mu_k^{(t)}, t)$, $p_3 = ((k, \mu_k^{(t)}+1, t)$ and $p_4 = (k+1, 1, t)$. Then

$$\begin{aligned}
&\prod_{\beta\in\mu_k^{(t)}} \frac{(y+c'(\beta))^2-1}{(y-c'(\beta))^2-1} \cdot \frac{(y-c'(\beta))^2}{(y+c'(\beta))^2}\\
&\quad= \prod_{\beta\in\mu_k^{(t)}} \frac{y-c'(\beta)}{y-(c'(\beta)+1)} \frac{y-c'(\beta)}{y-(c'(\beta)-1)} \frac{y+(c'(\beta)+1)}{y+c'(\beta)} \frac{y+(c'(\beta)-1)}{y+c'(\beta)}\\
&\quad= \prod_{\beta\in\mu_k^{(t)}} \frac{y-c'(p_1)}{y-c'(p_3)} \frac{y-c'(p_2)}{y-c'(p_4)} \frac{y+c'(p_3)}{y+c'(p_1)} \frac{y+c'(p_4)}{y+c'(p_2)}\\
&\quad= \prod_{\beta\in\mu_k^{(t)}} \frac{y-c'(p_1)}{y+c'(p_1)} \frac{y-c'(p_2)}{y+c'(p_2)} \frac{y+c'(p_3)}{y-c'(p_3)} \frac{y+c'(p_4)}{y-c'(p_4)}
\end{aligned}$$

where for $\beta = (a,b,t)$ we write $c'(\beta) = b - a + u_t$. Taking the product over all k shows that

$$\frac{(y+u_t)}{(y-u_t)} \prod_{\beta\in\mu^{(t)}} \frac{(y+c(\beta))^2-1}{(y-c(\beta))^2-1} \cdot \frac{(y-c(\beta))^2}{(y+c(\beta))^2} = \prod_{\alpha} \frac{y+c(\alpha)}{y-c(\alpha)},$$

where, in the first product, every node is considered to be an addable node and, in the second product, α runs over the addable and removable nodes of $\mu^{(t)}$. Hence,

$$\widetilde{W}_k(y,\mathfrak{t}) + y - \frac{1}{2} = \big(y - \frac{1}{2}(-1)^r\big) \prod_{\alpha} \frac{y+c(\alpha)}{y-c(\alpha)},$$

where α runs over the addable and removable nodes of $\mathfrak{t}_{k-1} = (\mu^{(1)}, \dots, \mu^{(r)})$. $\square$

Corollary 4.18. *Suppose that* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ *and that* $1 \le k < n$ *and* $a \ge 0$. *Then* $E_k X_k^a E_k v_\mathfrak{t} = \omega_k^{(a)} E_k v_\mathfrak{t}$.

Proof. If $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$ then $E_k X_k^i E_k v_\mathfrak{t} = 0 = \omega_k^{(i)} E_k v_\mathfrak{t}$, so we may assume that $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$. Now, by definition, $e_{\mathfrak{tu}}(k) = \sqrt{e_{\mathfrak{tt}}(k)}\sqrt{e_{\mathfrak{uu}}(k)}$. So

$$\begin{aligned}
E_k \frac{y}{y - X_k} E_k v_\mathfrak{t} &= E_k \sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \frac{y}{y - c_\mathfrak{u}(k)} e_{\mathfrak{tu}}(k) v_\mathfrak{u} = \sum_{\mathfrak{w} \overset{k}{\sim} \mathfrak{u}} \sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \frac{y}{y - c_\mathfrak{u}(k)} e_{\mathfrak{uw}}(k) e_{\mathfrak{tu}}(k) v_\mathfrak{w} \\
&= \sum_{\mathfrak{w} \overset{k}{\sim} \mathfrak{t}} \left(\sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \frac{y}{y - c_\mathfrak{u}(k)} e_{\mathfrak{uu}}(k) \right) e_{\mathfrak{tw}}(k) v_\mathfrak{w} \\
&= W_k(y, \mathfrak{t}) E_k v_\mathfrak{t} = \widetilde{W}_k(y, \mathfrak{t}) E_k v_\mathfrak{t},
\end{aligned}$$

by Proposition 4.17. By Lemma 4.15, $\omega_k^{(a)} \in R[X_1, \dots, X_{k-1}]$, so $\omega_k^{(a)} v_\mathfrak{t} = \omega_k^{(a)} v_\mathfrak{u}$ whenever $\mathfrak{t} \overset{k}{\sim} \mathfrak{u}$. Therefore,

$$\begin{aligned}
E_k \frac{y}{y - X_k} E_k v_\mathfrak{t} &= \sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} e_{\mathfrak{tu}}(k) \widetilde{W}_k(y, \mathfrak{t}) v_\mathfrak{u} = \sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} e_{\mathfrak{tu}}(k) \widetilde{W}_k(y, \mathfrak{u}) v_\mathfrak{u} \\
&= \widetilde{W}_k(y) \sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} e_{\mathfrak{tu}}(k) v_\mathfrak{u} = \widetilde{W}_k(y) E_k v_\mathfrak{t}.
\end{aligned}$$

Comparing the coefficient of y^{-a}, for $a \geq 0$, on both sides of the last equation proves the Corollary. □

Lemma 4.19. *Suppose that* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ *with* $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$ *and* $\mathfrak{t}_k = \mathfrak{t}_{k+2}$*. Then* $e_{\mathfrak{tt}}(k) e_{\mathfrak{tt}}(k+1) = 1$.

Proof. The recursion formula of Lemma 4.15 and Proposition 4.17 show that

$$W_{k+1}(y, \mathfrak{t}) + y - \frac{1}{2} = \big(W_k(y, \mathfrak{t}) + y - \frac{1}{2}\big) \frac{(y - c_\mathfrak{t}(k))^2}{(y + c_\mathfrak{t}(k))^2} \frac{(y + c_\mathfrak{t}(k))^2 - 1}{(y - c_\mathfrak{t}(k))^2 - 1},$$

and, by definition,

$$W_k(y, \mathfrak{t}) + y - \frac{1}{2} = \big(y - \frac{1}{2}(-1)^r\big) \prod_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \frac{y + c_\mathfrak{u}(k)}{y - c_\mathfrak{u}(k)}.$$

Thus,

$$\begin{aligned}
\frac{W_{k+1}(y, \mathfrak{t}) + y - \frac{1}{2}}{y} =& \Big(1 - \frac{1}{2y}(-1)^r\Big) \frac{y - c_\mathfrak{t}(k)}{y + c_\mathfrak{t}(k)} \frac{(y + c_\mathfrak{t}(k))^2 - 1}{(y - c_\mathfrak{t}(k))^2 - 1} \\
&\times \prod_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}, \mathfrak{u} \neq \mathfrak{t}} \frac{y + c_\mathfrak{u}(k)}{y - c_\mathfrak{u}(k)}.
\end{aligned}$$

Taking residues at $y = -c_\mathfrak{t}(k) = c_\mathfrak{t}(k+1)$ on both sides of this equation, we have

$$\begin{aligned}
e_{\mathfrak{tt}}(k+1) &= \frac{2c_\mathfrak{t}(k) + (-1)^r}{4c_\mathfrak{t}(k)^2 - 1} \prod_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}, \mathfrak{u} \neq \mathfrak{t}} \frac{c_\mathfrak{t}(k) - c_\mathfrak{u}(k)}{c_\mathfrak{t}(k) + c_\mathfrak{u}(k)} \\
&= \frac{1}{2c_\mathfrak{t}(k) - (-1)^r} \prod_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}, \mathfrak{u} \neq \mathfrak{t}} \frac{c_\mathfrak{t}(k) - c_\mathfrak{u}(k)}{c_\mathfrak{t}(k) + c_\mathfrak{u}(k)} = \frac{1}{e_{\mathfrak{tt}}(k)}.
\end{aligned}$$

where the last equality uses (4.8). □

We remark that the condition $\mathfrak{t}_k = \mathfrak{t}_{k+2}$ is needed in Lemma 4.19 only because $e_{\mathfrak{tt}}(k+1)$ is not defined without this assumption.

Lemma 4.20. *Fix an integer k with $1 \le k < n-1$ and suppose that $\mathfrak{t}, \mathfrak{u}, \mathfrak{w} \in \mathscr{T}_n^{ud}(\lambda)$ are updown λ–tableaux such that $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$, $\mathfrak{t}_k = \mathfrak{t}_{k+2}$, $\mathfrak{u} \overset{k+1}{\sim} \mathfrak{t}$, $\mathfrak{w} \overset{k}{\sim} \mathfrak{t}$ and that $S_k\mathfrak{u}$ and $S_{k+1}\mathfrak{w}$ are both defined with $S_k\mathfrak{u} = S_{k+1}\mathfrak{w}$. Then $b_\mathfrak{u}(k)^2 e_{\mathfrak{u}\mathfrak{u}}(k+1) = b_\mathfrak{w}(k+1)^2 e_{\mathfrak{w}\mathfrak{w}}(k)$.*

Proof. Let $\sigma = \mathfrak{t}_k \ominus \mathfrak{t}_{k-1}$ and $\tau = \mathfrak{u}_{k+1} \ominus \mathfrak{t}_k$. $S_k\mathfrak{u} = S_{k+1}\mathfrak{w}$ implies $\tau = \mathfrak{w}_k \ominus \mathfrak{t}_{k-1}$. Then, by (4.8),

$$e_{\mathfrak{w}\mathfrak{w}}(k) = (2c(\tau) - (-1)^r) \prod_\alpha \frac{c(\tau) + c(\alpha)}{c(\tau) - c(\alpha)},$$

where α runs over the addable and removable nodes of $\mathfrak{t}_{k-1} = \mathfrak{w}_{k-1}$ with $c(\alpha) \neq c(\tau)$ and, similarly,

$$e_{\mathfrak{u}\mathfrak{u}}(k+1) = (2c(\tau) - (-1)^r) \prod_\alpha \frac{c(\tau) + c(\alpha)}{c(\tau) - c(\alpha)},$$

where α runs over all addable and removable nodes of $\mathfrak{t}_k = \mathfrak{u}_k$ with $c(\alpha) \neq c(\tau)$. Taking $c(\tau) \neq 0$ into consideration, we have

$$e_{\mathfrak{w}\mathfrak{w}}(k) = \operatorname*{Res}_{y=c(\tau)} \frac{W_k(y,\mathfrak{t}) + y - \frac{1}{2}}{y},$$

and

$$e_{\mathfrak{u}\mathfrak{u}}(k+1) = \operatorname*{Res}_{y=c(\tau)} \frac{W_{k+1}(y,\mathfrak{t}) + y - \frac{1}{2}}{y}.$$

Further, by Lemma 4.15 and Proposition 4.17, we have

$$W_{k+1}(y,\mathfrak{t}) + y - \frac{1}{2} = \big(W_k(y,\mathfrak{t}) + y - \frac{1}{2}\big) \frac{(y + c(\sigma))^2 - 1}{(y - c(\sigma))^2 - 1} \frac{(y - c(\sigma))^2}{(y + c(\sigma))^2}.$$

It follows that

$$\frac{e_{\mathfrak{u}\mathfrak{u}}(k+1)}{e_{\mathfrak{w}\mathfrak{w}}(k)} = \frac{(c(\sigma) + c(\tau))^2 - 1}{(c(\sigma) + c(\tau))^2} \frac{(c(\tau) - c(\sigma))^2}{(c(\tau) - c(\sigma))^2 - 1} = \frac{b_\mathfrak{w}(k+1)^2}{b_\mathfrak{u}(k)^2},$$

where the last equality follows from the definitions because $(c_\mathfrak{u}(k), c_\mathfrak{u}(k+1), c_\mathfrak{u}(k+2)) = (c(\sigma), c(\tau), -c(\tau))$ and $(c_\mathfrak{w}(k), c_\mathfrak{w}(k+1), c_\mathfrak{w}(k+2)) = (c(\tau), -c(\tau), c(\sigma))$. □

The following combinatorial identities will be used in the proof of Theorem 4.13.

Proposition 4.21. *Suppose that $\mathfrak{t}, \mathfrak{u}' \in \mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$, $\mathfrak{t}_k \neq \mathfrak{t}_{k+2}$, $\mathfrak{u}' \overset{k}{\sim} \mathfrak{t}$ and $\mathfrak{u}' \neq \mathfrak{t}$, where $1 \le k < n-1$. Let $\tilde{\mathfrak{t}} \in \mathscr{T}_n^{ud}(\lambda)$ be the updown tableau which is uniquely determined by the conditions $\tilde{\mathfrak{t}} \overset{k}{\sim} \mathfrak{t}$ and $\tilde{\mathfrak{t}}_k = \mathfrak{t}_{k+2}$. Then the following identities hold:*

a) $\displaystyle\sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \frac{e_{\mathfrak{u}\mathfrak{u}}(k)}{c_\mathfrak{t}(k) + c_\mathfrak{u}(k)} = 1 + \frac{1}{2c_\mathfrak{t}(k)},$

b) $\displaystyle\sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \frac{e_{\mathfrak{u}\mathfrak{u}}(k)}{(c_\mathfrak{t}(k) + c_\mathfrak{u}(k))^2} = \Big(1 - \frac{1}{4c_\mathfrak{t}(k)^2}\Big) \frac{1}{e_{\mathfrak{t}\mathfrak{t}}(k)} + \frac{1}{2c_\mathfrak{t}(k)^2}$

c) $\displaystyle\sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \frac{e_{\mathfrak{u}\mathfrak{u}}(k)}{(c_\mathfrak{t}(k) + c_\mathfrak{u}(k))(c_\mathfrak{u}(k) + c_{\mathfrak{u}'}(k))} = \frac{1}{2c_\mathfrak{t}(k) c_{\mathfrak{u}'}(k)},$

Proof. It follows from (4.9) and Definition 4.7 that

$$\frac{W_k(y,\mathfrak{t})}{y} = \sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \frac{e_{\mathfrak{u}\mathfrak{u}}(k)}{y - c_\mathfrak{u}(k)}.$$

Evaluating both sides at $y = -c_{\mathfrak{t}}(k)$ and using (4.5) gives (a).

By Proposition 4.17 and Corollary 4.18 we have

$$E_k \frac{1}{(y - X_k)(v - X_k)} E_k v_{\mathfrak{t}} = \frac{1}{v - y}\Big(\frac{W_k(y,\mathfrak{t})}{y} - \frac{W_k(v,\mathfrak{t})}{v}\Big) E_k v_{\mathfrak{t}}.$$

Comparing the coefficients of $v_{\mathfrak{t}}$ on both sides of this equation we obtain

$$\sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \frac{e_{\mathfrak{u}\mathfrak{u}}(k)}{(y - c_{\mathfrak{u}}(k))(v - c_{\mathfrak{u}}(k))} = \frac{1}{v - y}\Big\{\frac{W_k(y,\mathfrak{t})}{y} - \frac{W_k(v,\mathfrak{t})}{v}\Big\}.$$

Setting $y = -c_{\mathfrak{t}}(k)$ we obtain

$$\begin{aligned}\sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \frac{e_{\mathfrak{u}\mathfrak{u}}(k)}{(c_{\mathfrak{t}}(k) + c_{\mathfrak{u}}(k))(v - c_{\mathfrak{u}}(k))} &= \frac{1}{v + c_{\mathfrak{t}}(k)}\Bigg\{\Big(\frac{W_k(v,\mathfrak{t})}{v} + 1 - \frac{1}{2v}\Big) + \Big(\frac{1}{2c_{\mathfrak{t}}(k)} + \frac{1}{2v}\Big)\Bigg\}\\ &= \frac{2v - (-1)^r}{2v(v + c_{\mathfrak{t}}(k))} \prod_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \frac{v + c_{\mathfrak{u}}(k)}{v - c_{\mathfrak{u}}(k)} + \frac{1}{2c_{\mathfrak{t}}(k) v}.\end{aligned}$$

Setting $v = -c_{\mathfrak{u}'}(k)$ gives (c). Now we set $v = -c_{\mathfrak{t}}(k)$. Then it gives

$$\sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \frac{e_{\mathfrak{u}\mathfrak{u}}(k)}{(c_{\mathfrak{t}}(k) + c_{\mathfrak{u}}(k))^2} = \frac{2c_{\mathfrak{t}}(k) + (-1)^r}{4c_{\mathfrak{t}}(k)^2} \prod_{\substack{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}\\ \mathfrak{u} \neq \mathfrak{t}}} \frac{c_{\mathfrak{t}}(k) - c_{\mathfrak{u}}(k)}{c_{\mathfrak{t}}(k) + c_{\mathfrak{u}}(k)} + \frac{1}{2c_{\mathfrak{t}}(k)^2}.$$

On the other hand, multiplying the reciprocal of (4.8) by $\big(1 - \frac{1}{4c_{\mathfrak{t}}(k)^2}\big)$ gives

$$\Big(1 - \frac{1}{4c_{\mathfrak{t}}(k)^2}\Big)\frac{1}{e_{\mathfrak{t}\mathfrak{t}}(k)} = \frac{2c_{\mathfrak{t}}(k) + (-1)^r}{4c_{\mathfrak{t}}(k)^2} \prod_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}, \mathfrak{u} \neq \mathfrak{t}} \frac{c_{\mathfrak{t}}(k) - c_{\mathfrak{u}}(k)}{c_{\mathfrak{t}}(k) + c_{\mathfrak{u}}(k)}.$$

Combining these two equations gives (b). □

We are now ready to start checking that the action of $\mathscr{W}_{r,n}(\mathbf{u})$ on $\Delta(\lambda)$ respects the relations of $\mathscr{W}_{r,n}(\mathbf{u})$. We break the proof into several lemmas and propositions.

Lemma 4.22. *Suppose* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$. *Then*

a) $E_i^2 v_{\mathfrak{t}} = \omega_0 E_i v_{\mathfrak{t}}$, *for* $1 \le i < n$.
b) $E_1 X_1^a E_1 v_{\mathfrak{t}} = \omega_a E_1 v_{\mathfrak{t}}$, *for* $a > 0$.
c) $(X_1 - u_1)(X_1 - u_2)\cdots(X_1 - u_r) v_{\mathfrak{t}} = 0$.
d) $X_i X_j v_{\mathfrak{t}} = X_j X_i v_{\mathfrak{t}}$ *for* $1 \le i, j \le n$.
e) $E_i(X_i + X_{i+1}) v_{\mathfrak{t}} = (X_i + X_{i+1}) E_i v_{\mathfrak{t}} = 0$, $1 \le i \le n-1$.
f) $(S_i X_i - X_{i+1} S_i) v_{\mathfrak{t}} = (E_i - 1) v_{\mathfrak{t}} = (X_i S_i - S_i X_{i+1}) v_{\mathfrak{t}}$, *for* $1 \le i \le n-1$
g) $E_k E_l v_{\mathfrak{t}} = E_l E_k v_{\mathfrak{t}}$ *if* $|k - l| > 1$.
h) $E_k X_l v_{\mathfrak{t}} = X_l E_k v_{\mathfrak{t}}$ *if* $l \neq k, k+1$.
i) $S_k X_l v_{\mathfrak{t}} = X_l S_k v_{\mathfrak{t}}$ *if* $l \neq k, k+1$.

Proof. As $\omega_0 = \omega_1^{(0)}$ and $\omega_a = \omega_1^{(a)}$ by Lemma 4.15, parts (a) and (b) have already been proved in Corollary 4.18. Parts (c)–(f) follow directly from the definitions of the actions. If $|k - l| > 1$ then taking $\mathfrak{u} = S_l \mathfrak{t}$ in (4.12)(b) shows that (g) holds. Assume now that $l \neq k, k+1$. If $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$ then $c_{S_k \mathfrak{t}}(l) = c_{\mathfrak{t}}(l)$. If $\mathfrak{u} \overset{k}{\sim} \mathfrak{t}$ then $c_{\mathfrak{u}}(l) = c_{\mathfrak{t}}(l)$. Combining the last two statements forces (h) and (i) to be true. □

Lemma 4.23. *Suppose* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$. *Then* $E_k E_{k\pm1} E_k v_{\mathfrak{t}} = E_k v_{\mathfrak{t}}$.

Proof. We only prove that $E_k E_{k+1} E_k v_{\mathfrak{t}} = E_k v_{\mathfrak{t}}$, since the argument for the case $E_k E_{k-1} E_k v_{\mathfrak{t}} = E_k v_{\mathfrak{t}}$ is almost identical.

We may assume $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$ since, otherwise, $E_k E_{k+1} E_k v_{\mathfrak{t}} = 0 = E_k v_{\mathfrak{t}}$. Let $\tilde{\mathfrak{t}}$ be the unique n–updown tableau such that $\tilde{\mathfrak{t}} \overset{k}{\sim} \mathfrak{t}$ and $\tilde{\mathfrak{t}}_k = \mathfrak{t}_{k+2}$. We have

$$E_k E_{k+1} E_k v_{\mathfrak{t}} = e_{\mathfrak{t}\tilde{\mathfrak{t}}}(k) e_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k+1) \sum_{\mathfrak{u} \overset{k}{\sim} \tilde{s}} e_{\tilde{\mathfrak{t}}\mathfrak{u}}(k) v_{\mathfrak{u}} = e_{\tilde{\mathfrak{t}}\mathfrak{t}}(k) e_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k+1) \sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} e_{\mathfrak{t}\mathfrak{u}}(k) v_{\mathfrak{u}}.$$

Hence, $E_k E_{k+1} E_k v_{\mathfrak{t}} = E_k v_{\mathfrak{t}}$ by Lemma 4.19. □

It still remains to check relations (a), (b)(i), (b)(ii), (d)(i) and (f) from Definition 2.1.

Lemma 4.24. *Suppose that* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$. *Then* $S_k^2 v_{\mathfrak{t}} = v_{\mathfrak{t}}$.

Proof. **Case 1.** $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$:
If $S_k\mathfrak{t}$ is not defined then $a_{\mathfrak{t}}(k) \in \{-1, 1\}$ and $b_{\mathfrak{t}}(k) = 0$, which implies $S_k^2 v_{\mathfrak{t}} = v_{\mathfrak{t}}$. If $S_k\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ then by the choice of the square roots in (4.12)(a) we have

$$S_k^2 v_{\mathfrak{t}} = \Big(a_{\mathfrak{t}}(k)^2 + b_{\mathfrak{t}}(k) b_{S_k\mathfrak{t}}(k)\Big) v_{\mathfrak{t}} + \Big(a_{\mathfrak{t}}(k) + a_{S_k\mathfrak{t}}(k)\Big) b_{\mathfrak{t}}(k) v_{S_k\mathfrak{t}} = v_{\mathfrak{t}}.$$

Case 2. $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$:

We have $S_k^2 v_{\mathfrak{t}} = \sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \Big(\sum_{\mathfrak{v} \overset{k}{\sim} \mathfrak{t}} s_{\mathfrak{t}\mathfrak{v}}(k) s_{\mathfrak{v}\mathfrak{u}}(k) \Big) v_{\mathfrak{u}}$. So, the coefficient of $v_{\mathfrak{t}}$ in $S_k^2 \mathfrak{v}_s$ is

$$\sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} s_{\mathfrak{t}\mathfrak{u}}(k) s_{\mathfrak{u}\mathfrak{t}}(k) = \sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} \frac{e_{\mathfrak{t}\mathfrak{t}}(k) e_{\mathfrak{u}\mathfrak{u}}(k)}{(c_{\mathfrak{t}}(k) + c_{\mathfrak{u}}(k))^2} - \frac{e_{\mathfrak{t}\mathfrak{t}}}{2c_{\mathfrak{t}}(k)^2} + \frac{1}{4c_{\mathfrak{t}}(k)^2} = 1,$$

where the last equality follows by rearranging Proposition 4.21(b). If $\mathfrak{u} \overset{k}{\sim} \mathfrak{t}$ and $\mathfrak{u} \neq \mathfrak{t}$ then the coefficient of $v_{\mathfrak{u}}$ in $S_k^2 v_{\mathfrak{t}}$ is

$$\begin{aligned}
\sum_{\mathfrak{v} \overset{k}{\sim} \mathfrak{t}} s_{\mathfrak{t}\mathfrak{v}}(k) s_{\mathfrak{v}\mathfrak{u}}(k) &= \sum_{\substack{\mathfrak{v} \sim \mathfrak{t} \\ \mathfrak{t} \neq \mathfrak{v} \neq \mathfrak{u}}} \frac{e_{\mathfrak{t}\mathfrak{v}}(k) e_{\mathfrak{v}\mathfrak{u}}(k)}{(c_{\mathfrak{t}}(k) + c_{\mathfrak{v}}(k))(c_{\mathfrak{v}}(k) + c_{\mathfrak{u}}(k))} \\
&\quad + \frac{(e_{\mathfrak{t}\mathfrak{t}}(k) - 1) e_{\mathfrak{t}\mathfrak{u}}(k)}{2c_{\mathfrak{t}}(k)(c_{\mathfrak{t}}(k) + c_{\mathfrak{u}}(k))} + \frac{(e_{\mathfrak{u}\mathfrak{u}}(k) - 1) e_{\mathfrak{t}\mathfrak{u}}(k)}{2c_{\mathfrak{u}}(k)(c_{\mathfrak{t}}(k) + c_{\mathfrak{u}}(k))} \\
&= e_{\mathfrak{t}\mathfrak{u}}(k) \Big(\sum_{\mathfrak{v} \overset{k}{\sim} \mathfrak{t}} \frac{e_{\mathfrak{v}\mathfrak{v}}(k)}{(c_{\mathfrak{t}}(k) + c_{\mathfrak{v}}(k))(c_{\mathfrak{v}}(k) + c_{\mathfrak{u}}(k))} - \frac{1}{2c_{\mathfrak{t}}(k) c_{\mathfrak{u}}(k)} \Big) \\
&= 0
\end{aligned}$$

by Proposition 4.21(c). Therefore, $S_k^2 v_{\mathfrak{t}} = v_{\mathfrak{t}}$. □

The next two Propositions prove that the action of $\mathscr{W}_{r,n}(\mathbf{u})$ on $\Delta(\lambda)$ respects the tangle relations 2.1(g).

Proposition 4.25. *For any* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$, $E_k S_k v_{\mathfrak{t}} = E_k v_{\mathfrak{t}} = S_k E_k v_{\mathfrak{t}}$.

Proof. Suppose that $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$. Then either $S_k\mathfrak{t}$ is not defined, or $S_k\mathfrak{t} \overset{k}{\sim} \mathfrak{t}$. In either case, we have $E_k S_k v_{\mathfrak{t}} = E_k v_{\mathfrak{t}} = S_k E_k v_{\mathfrak{t}} = 0$. Suppose $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$. Then

$$S_k E_k v_{\mathfrak{t}} = \sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} e_{\mathfrak{t}\mathfrak{u}}(k) S_k v_{\mathfrak{u}} = \sum_{\mathfrak{u}' \overset{k}{\sim} \mathfrak{u}} \sum_{\mathfrak{u} \overset{k}{\sim} \mathfrak{t}} s_{\mathfrak{u}\mathfrak{u}'}(k) e_{\mathfrak{t}\mathfrak{u}}(k) v_{\mathfrak{u}'}.$$

By Proposition 4.21(a), we have

$$\begin{aligned}\sum_{\mathfrak{u}\overset{k}{\sim}\mathfrak{t}} e_{\mathfrak{t}\mathfrak{u}}(k)s_{\mathfrak{u}\mathfrak{u}'}(k) &= \sum_{\mathfrak{u}\overset{k}{\sim}\mathfrak{t},\mathfrak{u}\neq\mathfrak{u}'} \frac{e_{\mathfrak{t}\mathfrak{u}}(k)e_{\mathfrak{u}\mathfrak{u}'}(k)}{c_{\mathfrak{u}}(k)+c_{\mathfrak{u}'}(k)} + e_{\mathfrak{t}\mathfrak{u}'}(k)\frac{e_{\mathfrak{u}'\mathfrak{u}'}(k)-1}{2c_{\mathfrak{u}'}(k)}\\ &= e_{\mathfrak{t}\mathfrak{u}'}(k)\Big(\sum_{\mathfrak{u}\overset{k}{\sim}\mathfrak{t}} \frac{e_{\mathfrak{u}\mathfrak{u}}(k)}{c_{\mathfrak{u}}(k)+c_{\mathfrak{u}'}(k)} - \frac{1}{2c_{\mathfrak{u}'}(k)}\Big)\\ &= e_{\mathfrak{t}\mathfrak{u}'}(k).\end{aligned}$$

Hence, $S_kE_kv_{\mathfrak{t}} = E_kv_{\mathfrak{t}}$. One can prove that $E_kS_kv_{\mathfrak{t}} = E_kv_{\mathfrak{t}}$ similarly. □

Proposition 4.26. *Suppose that* $\mathfrak{t}\in\mathscr{T}_n^{ud}(\lambda)$. *Then*

a) $S_kE_{k+1}E_kv_{\mathfrak{t}} = S_{k+1}E_kv_{\mathfrak{t}}$.

b) $E_{k+1}E_kS_{k+1}v_{\mathfrak{t}} = E_{k+1}S_kv_{\mathfrak{t}}$.

Proof. (a) We may assume that $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$ since otherwise $S_kE_{k+1}E_kv_{\mathfrak{t}} = S_{k+1}E_kv_{\mathfrak{t}} = 0$. Let $\tilde{\mathfrak{t}}\in\mathscr{T}_n^{ud}(\lambda)$ be the unique updown tableau such that $\tilde{\mathfrak{t}}\overset{k}{\sim}\mathfrak{t}$ and $\tilde{\mathfrak{t}}_k = \mathfrak{t}_{k+2}$. We have

$$\begin{aligned}S_kE_{k+1}E_kv_{\mathfrak{t}} &= e_{\mathfrak{t}\tilde{\mathfrak{t}}}(k)e_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k+1)\Big(s_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k)v_{\tilde{\mathfrak{t}}} + \sum_{\mathfrak{u}\overset{k}{\sim}\tilde{\mathfrak{t}},\mathfrak{u}\neq\tilde{\mathfrak{t}}} s_{\tilde{\mathfrak{t}}\mathfrak{u}}(k)v_{\mathfrak{u}}\Big)\\ &\quad + \sum_{\mathfrak{u}\overset{k+1}{\sim}\tilde{\mathfrak{t}},\mathfrak{u}\neq\tilde{\mathfrak{t}}} e_{\mathfrak{t}\tilde{\mathfrak{t}}}(k)e_{\tilde{\mathfrak{t}}\mathfrak{u}}(k+1)\Big(a_{\mathfrak{u}}(k)v_{\mathfrak{u}} + b_{\mathfrak{u}}(k)v_{S_k\mathfrak{u}}\Big).\end{aligned}$$

Observe that if $S_k\mathfrak{u}$ is defined, for $\mathfrak{u}$ in the second sum, then $(S_k\mathfrak{u}')_k\neq\mathfrak{t}_{k+2}$ and $\mathfrak{w} = S_{k+1}S_k\mathfrak{u}$ is also defined. Further, we have $\mathfrak{w}\overset{k}{\sim}\tilde{\mathfrak{t}}$ and $\mathfrak{w}\neq\mathfrak{t}$. Similarly,

$$\begin{aligned}S_{k+1}E_kv_{\mathfrak{t}} &= e_{\mathfrak{t}\tilde{\mathfrak{t}}}(k)\Big(s_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k+1)v_{\tilde{\mathfrak{t}}} + \sum_{\mathfrak{u}\overset{k+1}{\sim}\tilde{\mathfrak{t}},\mathfrak{u}\neq\tilde{\mathfrak{t}}} s_{\tilde{\mathfrak{t}}\mathfrak{u}}(k+1)v_{\mathfrak{u}}\Big)\\ &\quad + \sum_{\mathfrak{u}\overset{k}{\sim}\tilde{\mathfrak{t}},\mathfrak{u}\neq\tilde{\mathfrak{t}}} e_{\mathfrak{t}\mathfrak{u}}(k)\Big(a_{\mathfrak{u}}(k+1)v_{\mathfrak{u}} + b_{\mathfrak{u}}(k+1)v_{S_{k+1}\mathfrak{u}}\Big).\end{aligned}$$

We now compare the coefficients of $v_{\mathfrak{u}}$ in $S_kE_{k+1}E_kv_{\mathfrak{t}}$ and in $S_{k+1}E_kv_{\mathfrak{t}}$. First, observe that $e_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k)e_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k+1) = 1$ by Lemma 4.19.

Case 1. $\mathfrak{u} = \tilde{\mathfrak{t}}$:
Since $c_{\tilde{\mathfrak{t}}}(k) = -c_{\tilde{\mathfrak{t}}}(k+1)$, the definitions and the remarks above show that the coefficient of $v_{\mathfrak{u}}$ in $S_kE_{k+1}E_kv_{\mathfrak{t}}$ is equal to

$$e_{\mathfrak{t}\tilde{\mathfrak{t}}}(k)e_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k+1)s_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k) = e_{\mathfrak{t}\tilde{\mathfrak{t}}}(k)\frac{1-e_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k+1)}{2c_{\tilde{\mathfrak{t}}}(k)} = e_{\mathfrak{t}\tilde{\mathfrak{t}}}(k)s_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k+1),$$

which is the coefficient of $v_{\mathfrak{u}}$ in $S_{k+1}E_kv_{\mathfrak{t}}$.

Case 2. $\mathfrak{u}\overset{k}{\sim}\tilde{\mathfrak{t}}$ **and** $\mathfrak{u}\neq\tilde{\mathfrak{t}}$:
Now, $c_{\tilde{\mathfrak{t}}}(k) = c_{\mathfrak{u}}(k+2)$ and $c_{\mathfrak{u}}(k+1) = -c_{\mathfrak{u}}(k)$, so the coefficient of $v_{\mathfrak{u}}$ in $S_kE_{k+1}E_kv_{\mathfrak{t}}$ is

$$e_{\mathfrak{t}\tilde{\mathfrak{t}}}(k)e_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k+1)s_{\tilde{\mathfrak{t}}\mathfrak{u}}(k) = \frac{e_{\mathfrak{t}\mathfrak{u}}(k)}{c_{\tilde{\mathfrak{t}}}(k)+c_{\mathfrak{u}}(k)} = e_{\mathfrak{t}\mathfrak{u}}(k)a_{\mathfrak{u}}(k+1),$$

which is the coefficient of $v_{\mathfrak{u}}$ in $S_{k+1}E_kv_{\mathfrak{t}}$.

Case 3. $\mathfrak{u}\overset{k+1}{\sim}\tilde{\mathfrak{t}}$ **and** $\mathfrak{u}\neq\tilde{\mathfrak{t}}$:
Since $c_{\mathfrak{u}}(k) = -c_{\tilde{\mathfrak{t}}}(k+1)$, the coefficient of $v_{\mathfrak{u}}$ in $S_kE_{k+1}E_kv_{\mathfrak{t}}$ is

$$a_{\mathfrak{u}}(k)e_{\tilde{\mathfrak{t}}\mathfrak{u}}(k+1)e_{s\tilde{\mathfrak{t}}}(k) = \frac{e_{\tilde{\mathfrak{t}}\mathfrak{u}}(k+1)e_{s\tilde{\mathfrak{t}}}(k)}{c_{\mathfrak{u}}(k+1)+c_{\tilde{\mathfrak{t}}}(k+1)} = e_{\mathfrak{t}\tilde{\mathfrak{t}}}(k)s_{\tilde{\mathfrak{t}}\mathfrak{u}}(k+1),$$

which is the coefficient of $v_{\mathfrak{u}}$ in $S_{k+1}E_kv_{\mathfrak{t}}$.

Now suppose that $S_k\mathfrak{u}$ is defined and let $\mathfrak{w}=S_{k+1}S_k\mathfrak{u}$ be as above. Then the coefficient of $v_{S_k\mathfrak{u}}$ in $S_kE_{k+1}E_kv_{\mathfrak{t}}$ is

$$\begin{aligned}e_{\mathfrak{t}\tilde{\mathfrak{t}}}(k)e_{\tilde{\mathfrak{t}}\mathfrak{u}}(k+1)b_{\mathfrak{u}}(k)&=\sqrt{e_{\mathfrak{t}\mathfrak{t}}(k)}\sqrt{e_{\mathfrak{u}\mathfrak{u}}(k+1)}b_{\mathfrak{u}}(k)\\&=\sqrt{e_{\mathfrak{t}\mathfrak{t}}(k)}\sqrt{e_{\mathfrak{w}\mathfrak{w}}(k)}b_{\mathfrak{w}}(k+1)\\&=e_{\mathfrak{t}\mathfrak{w}}(k)b_{\mathfrak{w}}(k+1),\end{aligned}$$

where the second equality comes from (4.12)(f). As $S_k\mathfrak{u}=S_{k+1}\mathfrak{w}$ this is the coefficient of $v_{S_k\mathfrak{u}}$ in $S_{k+1}E_kv_{\mathfrak{t}}$. This completes the proof of (a).

(b) We let the reader work out the expansions of $E_{k+1}E_kS_{k+1}v_{\mathfrak{t}}$ and $E_{k+1}S_kv_{\mathfrak{t}}$. To show that these two expressions are equal there are four cases to consider.

Case 1. $\mathfrak{t}_k=\mathfrak{t}_{k+2}$ **and** $\mathfrak{t}_{k-1}=\mathfrak{t}_{k+1}$:
We have

$$\begin{aligned}E_{k+1}E_kS_{k+1}v_{\mathfrak{t}}&=E_{k+1}e_{\mathfrak{t}\mathfrak{t}}(k)s_{\mathfrak{t}\mathfrak{t}}(k+1)v_{\mathfrak{t}}=\frac{1-e_{\mathfrak{t}\mathfrak{t}}(k)}{2c_{\mathfrak{t}}(k+1)}E_{k+1}v_{\mathfrak{t}}\\&=s_{\mathfrak{t}\mathfrak{t}}(k)E_{k+1}v_{\mathfrak{t}}=E_{k+1}S_kv_{\mathfrak{t}}.\end{aligned}$$

Case 2. $\mathfrak{t}_k\ne\mathfrak{t}_{k+2}$ **and** $\mathfrak{t}_{k-1}=\mathfrak{t}_{k+1}$:
Define $\tilde{\mathfrak{t}}\in\mathscr{T}^{ud}_n(\lambda)$ to be the unique updown tableau such that $\tilde{\mathfrak{t}}\overset{k}{\sim}\mathfrak{t}$ and $\tilde{\mathfrak{t}}_k=\mathfrak{t}_{k+2}$. Then $\tilde{\mathfrak{t}}\ne\mathfrak{t}$ and

$$E_{k+1}E_kS_{k+1}v_{\mathfrak{t}}=a_{\mathfrak{t}}(k+1)e_{\mathfrak{t}\tilde{\mathfrak{t}}}(k)E_{k+1}v_{\tilde{\mathfrak{t}}}=s_{\mathfrak{t}\tilde{\mathfrak{t}}}(k)E_{k+1}v_{\tilde{\mathfrak{t}}}=E_{k+1}S_kv_s,$$

where the second equality uses the facts that $c_{\mathfrak{t}}(k+1)=-c_{\mathfrak{t}}(k)$, $c_{\mathfrak{t}}(k+2)=c_{\tilde{\mathfrak{t}}}(k)$ and $(S_{k+1}\mathfrak{t})_{k-1}\ne(S_{k+1}\mathfrak{t})_{k+1}$.

Case 3. $\mathfrak{t}_k=\mathfrak{t}_{k+2}$ **and** $\mathfrak{t}_{k-1}\ne\mathfrak{t}_{k+1}$:
Define $\tilde{\mathfrak{t}}\in\mathscr{T}^{ud}_n(\lambda)$ to be the unique updown tableau such that $\tilde{\mathfrak{t}}\overset{k+1}{\sim}\mathfrak{t}$ and $\tilde{\mathfrak{t}}_{k+1}=\mathfrak{t}_{k-1}$. Then

$$\begin{aligned}E_{k+1}E_kS_{k+1}v_{\mathfrak{t}}&=s_{\mathfrak{t}\tilde{\mathfrak{t}}}(k+1)e_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k)E_{k+1}v_{\tilde{\mathfrak{t}}}\\&=\frac{e_{\mathfrak{t}\tilde{\mathfrak{t}}}(k+1)e_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k)}{c_{\mathfrak{t}}(k+1)+c_{\tilde{\mathfrak{t}}}(k+1)}\sum_{\mathfrak{u}\overset{k+1}{\sim}\tilde{\mathfrak{t}}}e_{\tilde{\mathfrak{t}}\mathfrak{u}}(k+1)v_{\mathfrak{u}}\\&=a_{\mathfrak{t}}(k)\sum_{\tilde{\mathfrak{t}}\overset{k+1}{\sim}\mathfrak{u}}e_{\mathfrak{t}\mathfrak{u}}(k+1)v_{\mathfrak{u}}=E_{k+1}S_kv_{\mathfrak{t}},\end{aligned}$$

where we have used the facts that $c_{\tilde{\mathfrak{t}}}(k+1)=-c_{\mathfrak{t}}(k)$ and $(S_k\mathfrak{t})_k\ne(S_k\mathfrak{t})_{k+2}$.

Case 4. $\mathfrak{t}_k\ne\mathfrak{t}_{k+2}$ **and** $\mathfrak{t}_{k-1}\ne\mathfrak{t}_{k+1}$:
First observe that because of our assumptions we have $E_{k+1}E_kS_{k+1}v_{\mathfrak{t}}=b_{\mathfrak{t}}(k+1)E_{k+1}E_kv_{S_{k+1}\mathfrak{t}}$ and $E_{k+1}S_kv_{\mathfrak{t}}=b_{\mathfrak{t}}(k)E_{k+1}v_{S_k\mathfrak{t}}$. If $(S_{k+1}\mathfrak{t})_{k-1}\ne(S_{k+1}\mathfrak{t})_{k+1}$ then we also have $(S_k\mathfrak{t})_k\ne(S_k\mathfrak{t})_{k+2}$ so that $E_{k+1}S_kS_{k+1}v_{\mathfrak{t}}=0=E_{k+1}S_kv_{\mathfrak{t}}$.

Suppose now that $(S_{k+1}\mathfrak{t})_{k-1}=(S_{k+1}\mathfrak{t})_{k+1}$ and let $\tilde{\mathfrak{t}}\in\mathscr{T}^{ud}_n(\lambda)$ be the unique updown tableau such that $\tilde{\mathfrak{t}}\overset{k}{\sim}S_{k+1}\mathfrak{t}$ and $\tilde{\mathfrak{t}}_k=\mathfrak{t}_{k+2}$. Set $\mathfrak{u}=S_k\mathfrak{t}$ and $\mathfrak{w}=S_{k+1}\mathfrak{t}$ and observe that the assumptions of (4.12)(f) hold, so that $b_{\mathfrak{u}}(k)\sqrt{e_{\mathfrak{u}\mathfrak{u}}(k+1)}=b_{\mathfrak{w}}(k+1)\sqrt{e_{\mathfrak{w}\mathfrak{w}}(k)}$. As $b_{\mathfrak{t}}(k)=b_{\mathfrak{u}}(k)$ and $b_{\mathfrak{t}}(k+1)=b_{\mathfrak{w}}(k+1)$, the reader should now have no difficulty in using (4.12)(d), together with the fact that $\mathfrak{u}'\overset{k+1}{\sim}\tilde{\mathfrak{t}}$ if and only if $\mathfrak{u}' \overset{k+1}{\sim} S_k\mathfrak{t}$, to show that

$$\begin{aligned} E_{k+1}E_kS_{k+1}v_{\mathfrak{t}} &= b_{\mathfrak{t}}(k+1)\sum_{\mathfrak{u}'\overset{k+1}{\sim}\tilde{\mathfrak{t}}} e_{\tilde{\mathfrak{t}},S_{k+1}\mathfrak{t}}(k)e_{\tilde{\mathfrak{t}},\mathfrak{u}'}(k+1)v_{\mathfrak{u}'}\\ &= b_{\mathfrak{t}}(k)\sum_{\mathfrak{u}'\overset{k+1}{\sim}S_k\mathfrak{t}} e_{S_k\mathfrak{t},\mathfrak{u}'}(k+1)v_{\mathfrak{u}'} = E_{k+1}S_kv_{\mathfrak{t}}. \end{aligned}$$

□

The next Proposition shows that the action of $\mathscr{W}_{r,n}(\mathbf{u})$ on $\Delta(\lambda)$ respects the two relations 2.1(b)(i) and 2.1(d)(i).

Proposition 4.27. *Suppose that $\mathfrak{t}\in\mathscr{T}^{ud}_n(\lambda)$ and that $|k-l|>1$. Then:*
a) $S_kS_lv_{\mathfrak{t}} = S_lS_kv_{\mathfrak{t}}$.
b) $S_kE_lv_{\mathfrak{t}} = E_lS_kv_{\mathfrak{t}}$.

Proof. We prove only (a) as the proof of part (b) is similar to, but easier than (a).
First suppose that $\mathfrak{t}_{k-1}=\mathfrak{t}_{k+1}$ and $\mathfrak{t}_{l-1}=\mathfrak{t}_{l+1}$. Then

$$S_kS_lv_{\mathfrak{t}} = \sum_{\mathfrak{u}\overset{l}{\sim}\mathfrak{t},\ \mathfrak{w}\overset{k}{\sim}\mathfrak{u}} s_{\mathfrak{t}\mathfrak{u}}(l)s_{\mathfrak{u}\mathfrak{w}}(k)v_{\mathfrak{w}}.$$

Now for each pair of updown tableaux $(\mathfrak{w},\mathfrak{u})$ with $\mathfrak{w}\overset{k}{\sim}\mathfrak{u}\overset{l}{\sim}\mathfrak{t}$ there is a unique updown tableau $\mathfrak{u}'$ such that $\mathfrak{w}\overset{l}{\sim}\mathfrak{u}'\overset{k}{\sim}\mathfrak{t}$; more precisely, $\mathfrak{u}'_k=\mathfrak{w}_k$ and $\mathfrak{u}'_a=\mathfrak{t}_a$ for $a\ne l$. Notice that $\delta_{\mathfrak{u}\mathfrak{w}}=\delta_{\mathfrak{t}\mathfrak{u}'}$ and $\delta_{\mathfrak{t}\mathfrak{u}}=\delta_{\mathfrak{w}\mathfrak{u}'}$. Therefore,

$$\begin{aligned} s_{\mathfrak{t}\mathfrak{u}}(l)s_{\mathfrak{u}\mathfrak{w}}(k) &= \frac{\sqrt{e_{\mathfrak{t}\mathfrak{t}}(l)}\sqrt{e_{\mathfrak{u}\mathfrak{u}}(l)}-\delta_{\mathfrak{t}\mathfrak{u}}}{c_{\mathfrak{t}}(l)+c_{\mathfrak{u}}(l)}\,\frac{\sqrt{e_{\mathfrak{u}\mathfrak{u}}(k)}\sqrt{e_{\mathfrak{w}\mathfrak{w}}(k)}-\delta_{\mathfrak{u}\mathfrak{w}}}{c_{\mathfrak{u}}(k)+c_{\mathfrak{w}}(k)}\\ &= \frac{\sqrt{e_{\mathfrak{u}'\mathfrak{u}'}(l)}\sqrt{e_{\mathfrak{w}\mathfrak{w}}(l)}-\delta_{\mathfrak{w}\mathfrak{u}'}}{c_{\mathfrak{u}'}(l)+c_{\mathfrak{w}}(l)}\,\frac{\sqrt{e_{\mathfrak{t}\mathfrak{t}}(k)}\sqrt{e_{\mathfrak{u}'\mathfrak{u}'}(k)}-\delta_{\mathfrak{u}'\mathfrak{t}}}{c_{\mathfrak{t}}(k)+c_{\mathfrak{u}'}(k)}\\ &= s_{\mathfrak{u}'\mathfrak{w}}(l)s_{\mathfrak{t}\mathfrak{u}'}(k), \end{aligned}$$

where the second equality uses (4.8) and (4.12)(e). Hence,

$$S_kS_lv_{\mathfrak{t}} = \sum_{\mathfrak{u}\overset{l}{\sim}\mathfrak{t},\ \mathfrak{w}\overset{k}{\sim}\mathfrak{u}} s_{\mathfrak{t}\mathfrak{u}}(l)s_{\mathfrak{u}\mathfrak{w}}(k)v_{\mathfrak{w}} = \sum_{\mathfrak{u}'\overset{k}{\sim}\mathfrak{t},\ \mathfrak{w}\overset{l}{\sim}\mathfrak{u}} s_{\mathfrak{t}\mathfrak{u}'}(k)s_{\mathfrak{u}'\mathfrak{w}}(l)v_{\mathfrak{w}} = S_lS_kv_{\mathfrak{t}},$$

as required.
Assume now that $\mathfrak{t}_{k-1}\ne\mathfrak{t}_{k+1}$ and $\mathfrak{t}_{l-1}=\mathfrak{t}_{l+1}$. Then

$$\begin{aligned} S_kS_lv_{\mathfrak{t}} &= \sum_{\mathfrak{u}\overset{l}{\sim}\mathfrak{t}} s_{\mathfrak{t}\mathfrak{u}}(l)\big(a_{\mathfrak{u}}(k)v_{\mathfrak{u}}+b_{\mathfrak{u}}(k)v_{S_k\mathfrak{u}}\big)\\ &= a_{\mathfrak{t}}(k)\sum_{\mathfrak{u}\overset{l}{\sim}\mathfrak{t}} s_{\mathfrak{t}\mathfrak{u}}(l)v_{\mathfrak{u}} + b_{\mathfrak{t}}(k)\sum_{\mathfrak{u}\overset{l}{\sim}\mathfrak{t}} s_{\mathfrak{t}\mathfrak{u}}(l)v_{S_k\mathfrak{u}}\\ &= a_{\mathfrak{t}}(k)\sum_{\mathfrak{u}\overset{l}{\sim}\mathfrak{t}} s_{\mathfrak{t}\mathfrak{u}}(l)v_{\mathfrak{u}} + b_{\mathfrak{t}}(k)\sum_{\mathfrak{u}'\overset{l}{\sim}S_k\mathfrak{t}} s_{S_k\mathfrak{t},\mathfrak{u}'}(l)v_{\mathfrak{u}'} = S_lS_kv_{\mathfrak{t}}. \end{aligned}$$

Interchanging k and l covers the case when $\mathfrak{t}_{k-1}=\mathfrak{t}_{k+1}$ and $\mathfrak{t}_{l-1}\ne\mathfrak{t}_{l+1}$
Finally, consider the case when $\mathfrak{t}_{k-1}\ne\mathfrak{t}_{k+1}$ and $\mathfrak{t}_{l-1}\ne\mathfrak{t}_{l+1}$. Then

$$\begin{aligned} S_kS_lv_{\mathfrak{t}} &= a_{\mathfrak{t}}(k)a_{\mathfrak{t}}(l)v_{\mathfrak{t}} + a_{S_l\mathfrak{t}}(k)b_{\mathfrak{t}}(l)v_{S_l\mathfrak{t}} + b_{\mathfrak{t}}(k)a_{\mathfrak{t}}(l)v_{S_k\mathfrak{t}} + b_{S_l\mathfrak{t}}(k)b_{\mathfrak{t}}(l)v_{S_kS_l\mathfrak{t}}\\ &= a_{\mathfrak{t}}(l)a_{\mathfrak{t}}(k)v_{\mathfrak{t}} + a_{\mathfrak{t}}(k)b_{\mathfrak{t}}(l)v_{S_l\mathfrak{t}} + a_{S_k\mathfrak{t}}(l)b_{\mathfrak{t}}(k)v_{S_k\mathfrak{t}} + b_{S_k\mathfrak{t}}(l)b_{\mathfrak{t}}(k)v_{S_lS_k\mathfrak{t}}, \end{aligned}$$

since $a_{S_l\mathfrak{t}}(k)=a_{\mathfrak{t}}(k)$ and $a_{\mathfrak{t}}(l)=a_{S_k\mathfrak{t}}(l)$, by definition, and $b_{S_l\mathfrak{t}}(k)=b_{\mathfrak{t}}(k)$ and $b_{S_k\mathfrak{t}}(l)=b_{\mathfrak{t}}(l)$ by (4.12)(b). Hence, $S_kS_lv_{\mathfrak{t}}=S_lS_kv_{\mathfrak{t}}$ if $\mathfrak{t}_{k-1}\ne\mathfrak{t}_{k+1}$ and $\mathfrak{t}_{l-1}\ne\mathfrak{t}_{l+1}$. This completes the proof of (a). □

Finally, we prove that the action of $\mathscr{W}_{r,n}(\mathbf{u})$ on $\Delta(\lambda)$ respects the braid relations of length three.

Lemma 4.28. *Suppose that* $\mathfrak{t} \in \mathscr{T}^{ud}_n(\lambda)$ *with* $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$ *and* $\mathfrak{t}_k \neq \mathfrak{t}_{k+2}$*, where* $1 \le k < n-1$*. Then* $S_kS_{k+1}S_kv_\mathfrak{t} = S_{k+1}S_kS_{k+1}v_\mathfrak{t}$.

Proof. We consider two cases.

Case 1. $S_k\mathfrak{t}$ **is not defined, or** $S_k\mathfrak{t}$ **is defined and** $(S_k\mathfrak{t})_k \neq (S_k\mathfrak{t})_{k+2}$:
First suppose that $S_k\mathfrak{t}$ is defined. If $S_{k+1}\mathfrak{t}$ is defined then $(S_{k+1}\mathfrak{t})_{k-1} \neq (S_{k+1}\mathfrak{t})_{k+1}$, and if $S_{k+1}S_k\mathfrak{t}$ is defined then $(S_{k+1}S_k\mathfrak{t})_{k-1} \neq (S_{k+1}S_k\mathfrak{t})_{k+1}$ because $\mathfrak{t}_k \neq \mathfrak{t}_{k+2}$. Thus we have

$$\begin{aligned}
S_kS_{k+1}S_kv_\mathfrak{t} &= \big(a_\mathfrak{t}(k)^2a_\mathfrak{t}(k+1) + b_\mathfrak{t}(k)a_{S_k\mathfrak{t}}(k+1)b_{S_k\mathfrak{t}}(k)\big)v_\mathfrak{t}\\
&\quad+\big(a_\mathfrak{t}(k)a_\mathfrak{t}(k+1)b_\mathfrak{t}(k) + b_\mathfrak{t}(k)a_{S_k\mathfrak{t}}(k+1)a_{S_k\mathfrak{t}}(k)\big)v_{S_k\mathfrak{t}}\\
&\quad+a_\mathfrak{t}(k)b_\mathfrak{t}(k+1)a_{S_{k+1}\mathfrak{t}}(k)v_{S_{k+1}\mathfrak{t}} + a_\mathfrak{t}(k)b_\mathfrak{t}(k+1)b_{S_{k+1}\mathfrak{t}}(k)v_{S_kS_{k+1}\mathfrak{t}}\\
&\quad+b_\mathfrak{t}(k)b_{S_k\mathfrak{t}}(k+1)a_{S_{k+1}S_k\mathfrak{t}}(k)v_{S_{k+1}S_k\mathfrak{t}} + b_\mathfrak{t}(k)b_{S_k\mathfrak{t}}(k+1)b_{S_{k+1}S_k\mathfrak{t}}(k)v_{S_kS_{k+1}S_k\mathfrak{t}}
\end{aligned}$$

Now, $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$, or if $S_kS_{k+1}\mathfrak{t}$ is defined, then $(S_kS_{k+1}\mathfrak{t})_k \neq (S_kS_{k+1}\mathfrak{t})_{k+2}$. Therefore, we have

$$\begin{aligned}
S_{k+1}S_kS_{k+1}v_\mathfrak{t} &= \big(a_\mathfrak{t}(k+1)^2a_\mathfrak{t}(k) + b_\mathfrak{t}(k+1)a_{S_{k+1}\mathfrak{t}}(k)b_{S_{k+1}\mathfrak{t}}(k+1)\big)v_\mathfrak{t}\\
&\quad+\big(a_\mathfrak{t}(k+1)a_\mathfrak{t}(k)b_\mathfrak{t}(k+1) + b_\mathfrak{t}(k+1)a_{S_{k+1}\mathfrak{t}}(k)a_{S_{k+1}\mathfrak{t}}(k+1)\big)v_{S_{k+1}\mathfrak{t}}\\
&\quad+a_\mathfrak{t}(k+1)b_\mathfrak{t}(k)a_{S_k\mathfrak{t}}(k+1)v_{S_k\mathfrak{t}} + a_\mathfrak{t}(k+1)b_\mathfrak{t}(k)b_{S_k\mathfrak{t}}(k+1)v_{S_{k+1}S_k\mathfrak{t}}\\
&\quad+b_\mathfrak{t}(k+1)b_{S_{k+1}\mathfrak{t}}(k)a_{S_kS_{k+1}\mathfrak{t}}(k+1)v_{S_kS_{k+1}\mathfrak{t}}\\
&\quad+b_\mathfrak{t}(k+1)b_{S_{k+1}\mathfrak{t}}(k)b_{S_kS_{k+1}\mathfrak{t}}(k+1)v_{S_{k+1}S_kS_{k+1}\mathfrak{t}}
\end{aligned}$$

Now, $b_{S_k\mathfrak{t}}(k) = b_\mathfrak{t}(k)$ and $b_{S_{k+1}\mathfrak{t}}(k+1) = b_\mathfrak{t}(k+1)$ by (4.12)(a). So, in order to check that the coefficients of $v_\mathfrak{t}$ are equal in the last two equations we have to show that

$$a_\mathfrak{t}(k)^2a_\mathfrak{t}(k+1)+a_{S_k\mathfrak{t}}(k+1)(1-a_\mathfrak{t}(k)^2) = a_\mathfrak{t}(k)a_\mathfrak{t}(k+1)^2+a_{S_{k+1}\mathfrak{t}}(k+1)(1-a_\mathfrak{t}(k+1)^2);$$

however, this is just a special case of the easy identity

$$\frac{1}{(b-a)^2(c-b)} + \frac{1}{c-a}\Big(1-\frac{1}{(b-a)^2}\Big) = \frac{1}{(b-a)(c-b)^2} + \frac{1}{c-a}\Big(1-\frac{1}{(c-b)^2}\Big).$$

To see that the coefficients of $v_{S_k\mathfrak{t}}$ and $v_{S_{k+1}\mathfrak{t}}$ are equal amounts to the following easily checked identities

$$\begin{aligned}
a_{S_k\mathfrak{t}}(k)a_{S_k\mathfrak{t}}(k+1) + a_\mathfrak{t}(k)a_\mathfrak{t}(k+1) &= a_\mathfrak{t}(k+1)a_{S_k\mathfrak{t}}(k+1),\\
a_{S_{k+1}\mathfrak{t}}(k)a_{S_{k+1}\mathfrak{t}}(k+1) + a_\mathfrak{t}(k)a_\mathfrak{t}(k+1) &= a_\mathfrak{t}(k)a_{S_{k+1}\mathfrak{t}}(k).
\end{aligned}$$

For the coefficients of $v_{S_{k+1}S_k\mathfrak{t}}$ and $v_{S_kS_{k+1}\mathfrak{t}}$, note that $a_{S_{k+1}S_k\mathfrak{t}}(k) = a_\mathfrak{t}(k+1)$ and $a_{S_kS_{k+1}\mathfrak{t}}(k+1) = a_\mathfrak{t}(k)$. Finally, three applications of (4.12)(c) shows that the coefficients in $v_{S_kS_{k+1}S_k\mathfrak{t}} = v_{S_{k+1}S_kS_{k+1}\mathfrak{t}}$ are equal in both equations.

If $S_k\mathfrak{t}$ is not defined then $a_\mathfrak{t}(k) = \pm 1$ and $b_\mathfrak{t}(k) = 0$ by Lemma 4.11(b). Hence, the argument above is still valid if we set $b_\mathfrak{t}(k) = 0$.

Case 2. $S_k\mathfrak{t}$ **is defined and** $(S_k\mathfrak{t})_k = (S_k\mathfrak{t})_{k+2}$:
If $S_{k+1}\mathfrak{t}$ is defined then $(S_{k+1}\mathfrak{t})_{k-1} = (S_{k+1}\mathfrak{t})_{k+1}$. Let $\tilde{\mathfrak{t}}$ be the unique updown tableau such that $\tilde{\mathfrak{t}} \overset{k+1}{\sim} S_k\mathfrak{t}$ and $\tilde{\mathfrak{t}}_{k+1} = \mathfrak{t}_{k-1}$. Observe that if $\mathfrak{u} \overset{k+1}{\sim} \tilde{\mathfrak{t}}$ and $\mathfrak{u} \neq \tilde{\mathfrak{t}}$ then

$\mathfrak{u}_{k-1} \neq \mathfrak{u}_{k+1}$. Therefore,

$$\begin{aligned} S_kS_{k+1}S_kv_{\mathfrak{t}} &= a_{\mathfrak{t}}(k)^2a_{\mathfrak{t}}(k+1)v_{\mathfrak{t}} + a_{\mathfrak{t}}(k)a_{\mathfrak{t}}(k+1)b_{\mathfrak{t}}(k)v_{S_k\mathfrak{t}} \\ &\quad + a_{\mathfrak{t}}(k)b_{\mathfrak{t}}(k+1)\sum_{\mathfrak{u}\overset{k}{\sim}\tilde{\mathfrak{t}}} s_{S_{k+1}\mathfrak{t},\mathfrak{u}}(k)v_{\mathfrak{u}} + b_{\mathfrak{t}}(k)\sum_{\mathfrak{u}\overset{k}{\sim}\tilde{\mathfrak{t}}} s_{S_k\mathfrak{t},\tilde{\mathfrak{t}}}(k+1)s_{\tilde{\mathfrak{t}}\mathfrak{u}}(k)v_{\mathfrak{u}} \\ &\quad + \sum_{\substack{\mathfrak{u}\overset{k+1}{\sim}S_k\mathfrak{t}\\ \mathfrak{u}\neq\tilde{\mathfrak{t}}}} b_{\mathfrak{t}}(k)s_{S_k\mathfrak{t},\mathfrak{u}}(k+1)\Big(a_{\mathfrak{u}}(k)v_{\mathfrak{u}} + b_{\mathfrak{u}}(k)v_{S_k\mathfrak{u}}\Big), \end{aligned}$$

Similarly,

$$\begin{aligned} S_{k+1}S_kS_{k+1}v_{\mathfrak{t}} &= a_{\mathfrak{t}}(k+1)^2a_{\mathfrak{t}}(k)v_{\mathfrak{t}} + a_{\mathfrak{t}}(k+1)a_{\mathfrak{t}}(k)b_{\mathfrak{t}}(k+1)v_{S_{k+1}\mathfrak{t}} \\ &\quad + a_{\mathfrak{t}}(k+1)b_{\mathfrak{t}}(k)\sum_{\mathfrak{u}\overset{k+1}{\sim}\tilde{\mathfrak{t}}} s_{S_k\mathfrak{t},\mathfrak{u}}(k+1)v_{\mathfrak{u}} + b_{\mathfrak{t}}(k+1)\sum_{\mathfrak{u}\overset{k+1}{\sim}\tilde{\mathfrak{t}}} s_{S_{k+1}\mathfrak{t},\tilde{\mathfrak{t}}}(k)s_{\tilde{\mathfrak{t}}\mathfrak{u}}(k+1)v_{\mathfrak{u}} \\ &\quad + \sum_{\substack{\mathfrak{u}\overset{k}{\sim}S_{k+1}\mathfrak{t}\\ \mathfrak{u}\neq\tilde{\mathfrak{t}}}} b_{\mathfrak{t}}(k+1)s_{S_{k+1}\mathfrak{t},\mathfrak{u}}(k)\big(a_{\mathfrak{u}}(k+1)v_{\mathfrak{u}} + b_{\mathfrak{u}}(k+1)v_{S_{k+1}\mathfrak{u}}\big). \end{aligned}$$

We now compare each of the coefficients in the last two displayed equations.

First we consider the coefficient of $v_{\mathfrak{t}}$. To show that the coefficients of $v_{\mathfrak{t}}$ are equal in the two expressions above, we have to prove that

$$\begin{aligned} a_{\mathfrak{t}}(k)^2a_{\mathfrak{t}}(k+1) &+ b_{\mathfrak{t}}(k)s_{S_k\mathfrak{t},S_k\mathfrak{t}}(k+1)b_{S_k\mathfrak{t}}(k) \\ &= a_{\mathfrak{t}}(k+1)^2a_{\mathfrak{t}}(k) + b_{\mathfrak{t}}(k+1)s_{S_{k+1}\mathfrak{t},S_{k+1}\mathfrak{t}}(k)b_{S_{k+1}\mathfrak{t}}(k+1). \end{aligned}$$

Now, $b_{\mathfrak{t}}(k) = b_{S_k\mathfrak{t}}(k)$ and $b_{\mathfrak{t}}(k+1) = b_{S_{k+1}\mathfrak{t}}(k+1)$ by (4.12)(a). So, the last identity is equivalent to

$$\begin{aligned} a_{\mathfrak{t}}(k)^2a_{\mathfrak{t}}(k+1) &+ \frac{e_{S_k\mathfrak{t},S_k\mathfrak{t}}(k+1)-1}{2c_{S_k\mathfrak{t}}(k+1)}b_{S_k\mathfrak{t}}(k)^2 \\ &= a_{\mathfrak{t}}(k+1)^2a_{\mathfrak{t}}(k) + \frac{e_{S_{k+1}\mathfrak{t},S_{k+1}\mathfrak{t}}(k)-1}{2c_{S_{k+1}\mathfrak{t}}(k)}b_{S_{k+1}\mathfrak{t}}(k+1)^2. \end{aligned}$$

This equation is easily verified using the definitions and Lemma 4.20. Hence, the coefficients of $v_{\mathfrak{t}}$ in $S_kS_{k+1}S_kv_{\mathfrak{t}}$ and $S_{k+1}S_kS_{k+1}$ are equal.

Now consider the coefficient of $v_{S_k\mathfrak{t}}$ in both equations. Since $a_{S_k\mathfrak{t}}(k) - a_{\mathfrak{t}}(k+1) = 2c_{S_k\mathfrak{t}}(k+1)/(c_{\tilde{\mathfrak{t}}}(k) + c_{S_{k+1}\mathfrak{t}}(k))(c_{\tilde{\mathfrak{t}}}(k) + c_{S_k\mathfrak{t}}(k))$, we see that

$$\begin{aligned} s_{S_k\mathfrak{t},S_k\mathfrak{t}}(k+1)&(a_{S_k\mathfrak{t}}(k) - a_{\mathfrak{t}}(k+1))b_{\mathfrak{t}}(k) + a_{\mathfrak{t}}(k)a_{\mathfrak{t}}(k+1)b_{\mathfrak{t}}(k) \\ &= e_{S_k\mathfrak{t},S_k\mathfrak{t}}(k+1)a_{\mathfrak{t}}(k)a_{\mathfrak{t}}(k+1)b_{\mathfrak{t}}(k) \\ &= \frac{b_{S_k\mathfrak{t}}(k)e_{S_k\mathfrak{t},S_k\mathfrak{t}}(k+1)}{(c_{\tilde{\mathfrak{t}}}(k) + c_{S_{k+1}\mathfrak{t}}(k))(c_{\tilde{\mathfrak{t}}}(k+1) + c_{S_k\mathfrak{t}}(k+1))} \\ &= \frac{b_{S_{k+1}\mathfrak{t}}(k+1)\sqrt{e_{S_{k+1}\mathfrak{t},S_{k+1}\mathfrak{t}}(k)}\sqrt{e_{S_k\mathfrak{t},S_k\mathfrak{t}}(k+1)}}{(c_{\tilde{\mathfrak{t}}}(k) + c_{S_{k+1}\mathfrak{t}}(k))(c_{\tilde{\mathfrak{t}}}(k+1) + c_{S_k\mathfrak{t}}(k+1))}, \\ &= b_{\mathfrak{t}}(k+1)s_{S_{k+1}\mathfrak{t},\tilde{\mathfrak{t}}}(k)s_{\tilde{\mathfrak{t}},S_k\mathfrak{t}}(k+1). \end{aligned}$$

where the second last equality uses (4.12)(f). Consequently,

$$\begin{aligned} a_{\mathfrak{t}}(k)a_{\mathfrak{t}}(k+1)b_{\mathfrak{t}}(k) &+ b_{\mathfrak{t}}(k)s_{S_k\mathfrak{t},S_k\mathfrak{t}}(k+1)a_{S_k\mathfrak{t}}(k) \\ &= a_{\mathfrak{t}}(k+1)b_{\mathfrak{t}}(k)s_{S_k\mathfrak{t},S_k\mathfrak{t}}(k+1) + b_{\mathfrak{t}}(k+1)s_{S_{k+1}\mathfrak{t},\tilde{\mathfrak{t}}}(k)s_{\tilde{\mathfrak{t}},S_k\mathfrak{t}}(k+1) \end{aligned}$$

Hence, the coefficients of $v_{S_k\mathfrak{t}}$ in $S_kS_{k+1}S_kv_{\mathfrak{t}}$ and $S_{k+1}S_kS_{k+1}$ are equal. A similar argument shows that

$$\begin{aligned} a_{\mathfrak{t}}(k)b_{\mathfrak{t}}(k+1)s_{S_{k+1}\mathfrak{t},S_{k+1}\mathfrak{t}}(k) &+ b_{\mathfrak{t}}(k)s_{S_k\mathfrak{t},\tilde{\mathfrak{t}}}(k+1)s_{\tilde{\mathfrak{t}},S_{k+1}\mathfrak{t}}(k) \\ &= a_{\mathfrak{t}}(k+1)a_{\mathfrak{t}}(k)b_{\mathfrak{t}}(k+1) + b_{\mathfrak{t}}(k+1)s_{S_{k+1}\mathfrak{t},S_{k+1}\mathfrak{t}}(k)a_{S_{k+1}\mathfrak{t}}(k+1). \end{aligned}$$

This proves that the coefficient of $v_{S_{k+1}\mathfrak{t}}$ is the same in $v_{S_k\mathfrak{t}}$ in $S_kS_{k+1}S_kv_\mathfrak{t}$ and in $S_{k+1}S_kS_{k+1}$ are equal.

Now consider the coefficient of $v_\mathfrak{u}$ where $\mathfrak{u} \overset{k}{\sim} \tilde{\mathfrak{t}}$ and $\mathfrak{u} \notin \{\tilde{\mathfrak{t}}, S_{k+1}\mathfrak{t}\}$. This time

$$a_\mathfrak{u}(k+1) - a_\mathfrak{t}(k) = \frac{c_{S_{k+1}\mathfrak{t}}(k) + c_\mathfrak{u}(k)}{(c_{S_k\mathfrak{t}}(k+1) + c_{\tilde{\mathfrak{t}}}(k+1))(c_{\tilde{\mathfrak{t}}}(k) + c_\mathfrak{u}(k))}.$$

An argument similar to that for $v_{S_k\mathfrak{t}}$ now shows that

$$b_\mathfrak{t}(k)s_{S_k\mathfrak{t},\tilde{\mathfrak{t}}}(k+1)s_{\tilde{\mathfrak{t}}\mathfrak{u}}(k) = b_\mathfrak{t}(k+1)s_{S_{k+1}\mathfrak{t},\mathfrak{u}}(k)\big(a_\mathfrak{u}(k+1) - a_\mathfrak{t}(k)\big).$$

Therefore, the coefficients of $v_\mathfrak{u}$ for such $\mathfrak{u}$ in $S_kS_{k+1}S_kv_\mathfrak{t}$ and $S_{k+1}S_kS_{k+1}v_\mathfrak{t}$ are equal.

Another variation of this argument shows that if $\mathfrak{u} \overset{k+1}{\sim} S_k\mathfrak{t}$ and $\mathfrak{u} \notin \{\tilde{\mathfrak{t}}, S_k\mathfrak{t}\}$ then the coefficients of $v_\mathfrak{u}$ in $S_kS_{k+1}S_kv_\mathfrak{t}$ and $S_{k+1}S_kS_{k+1}$ are both equal.

Next, we suppose that $S_k\mathfrak{u}$ is defined and we compare the coefficients of $v_{S_k\mathfrak{u}}$ in $S_kS_{k+1}S_kv_\mathfrak{t}$ and $S_{k+1}S_kS_{k+1}v_\mathfrak{t}$, when $\mathfrak{u} \overset{k+1}{\sim} S_k\mathfrak{t}$ and $\mathfrak{u} \notin \{\tilde{\mathfrak{t}}, S_k\mathfrak{t}\}$. As $S_k\mathfrak{t}$ is defined, $\mathfrak{w} = S_{k+1}S_k\mathfrak{u}$ is defined and $\mathfrak{w} \overset{k}{\sim} S_{k+1}\mathfrak{t}$ with $\mathfrak{w} \notin \{\tilde{\mathfrak{t}}, S_{k+1}\mathfrak{t}\}$. Conversely, if $S_{k+1}\mathfrak{w}$ is defined for such $\mathfrak{w}$ then $\mathfrak{u} = S_kS_{k+1}\mathfrak{w}$ is defined. Applying (4.12)(f) twice, we have

$$b_\mathfrak{t}(k)b_\mathfrak{u}(k)\sqrt{e_{S_k\mathfrak{t},S_k\mathfrak{t}}(k+1)}\sqrt{e_{\mathfrak{u}\mathfrak{u}}(k+1)} = b_\mathfrak{t}(k{+}1)b_\mathfrak{w}(k{+}1)\sqrt{e_{S_{k+1}\mathfrak{t},S_{k+1}\mathfrak{t}}(k)}\sqrt{e_{\mathfrak{w}\mathfrak{w}}(k)}.$$

Consequently, because $c_{S_k\mathfrak{t}}(k+1) + c_\mathfrak{u}(k+1) = c_\mathfrak{w}(k) + c_{S_{k+1}\mathfrak{t}}(k)$, we have

$$b_\mathfrak{t}(k)s_{S_k\mathfrak{t},\mathfrak{u}}(k+1)b_\mathfrak{u}(k) = b_\mathfrak{t}(k+1)s_{S_{k+1}\mathfrak{t},\mathfrak{w}}(k)b_\mathfrak{w}(k+1).$$

That is, the coefficients of $v_{S_k\mathfrak{u}}$ in $S_{k+1}S_kS_{k+1}v_\mathfrak{t}$ and $S_kS_{k+1}S_kv_\mathfrak{t}$ are equal.

It remains to compare the coefficients of $v_{\tilde{\mathfrak{t}}}$ in the two equations. To show that these two coefficients are equal we have to prove that

$$\begin{aligned}a_\mathfrak{t}(k)b_\mathfrak{t}(k+1)s_{S_{k+1}\mathfrak{t},\tilde{\mathfrak{t}}}(k) &+ b_\mathfrak{t}(k)s_{S_k\mathfrak{t},\tilde{\mathfrak{t}}}(k+1)s_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k)\\ &= a_\mathfrak{t}(k+1)b_\mathfrak{t}(k)s_{S_k\mathfrak{t},\tilde{\mathfrak{t}}}(k+1) + b_\mathfrak{t}(k+1)s_{S_{k+1}\mathfrak{t},\tilde{\mathfrak{t}}}(k)s_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k+1).\end{aligned}$$

First note that, by the definitions and (4.12)(a),

$$\begin{aligned}b_\mathfrak{t}(k+1)s_{S_{k+1}\mathfrak{t},\tilde{\mathfrak{t}}}(k) &= \frac{b_\mathfrak{t}(k+1)\sqrt{e_{S_{k+1}\mathfrak{t},S_{k+1}\mathfrak{t}}(k)}\sqrt{e_{\tilde{\mathfrak{t}},\tilde{\mathfrak{t}}}(k)}}{c_{S_{k+1}\mathfrak{t}}(k) + c_{\tilde{\mathfrak{t}}}(k)}\\ &= \frac{b_\mathfrak{t}(k)\sqrt{e_{S_{k+1}\mathfrak{t},S_{k+1}\mathfrak{t}}(k)}\sqrt{e_{\tilde{\mathfrak{t}},\tilde{\mathfrak{t}}}(k+1)}}{c_{S_{k+1}\mathfrak{t}}(k) + c_{\tilde{\mathfrak{t}}}(k)}\\ &= b_\mathfrak{t}(k)s_{S_k\mathfrak{t},\tilde{\mathfrak{t}}}(k+1)\frac{c_{S_k\mathfrak{t}}(k+1) + c_{\tilde{\mathfrak{t}}}(k+1)}{c_{S_{k+1}\mathfrak{t}}(k) + c_{\tilde{\mathfrak{t}}}(k)}e_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k).\end{aligned}$$

So, it is enough to show that

$$\big(c_{S_k\mathfrak{t}}(k{+}1){+}c_{\tilde{\mathfrak{t}}}(k{+}1)\big)e_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k)\big(a_\mathfrak{t}(k){-}s_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k{+}1)\big) = \big(c_{S_{k+1}\mathfrak{t}}(k){+}c_{\tilde{\mathfrak{t}}}(k)\big)\big(a_\mathfrak{t}(k{+}1){-}s_{\tilde{\mathfrak{t}}\tilde{\mathfrak{t}}}(k)\big);$$

however, this follows from Lemma 4.19. Hence, the coefficients of $v_{\tilde{\mathfrak{t}}}$ in $S_{k+1}S_kS_{k+1}v_\mathfrak{t}$ and $S_kS_{k+1}S_kv_\mathfrak{t}$ are equal.

This completes the proof of Lemma 4.28. □

Lemma 4.29. *Suppose that* $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ *and that either* $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$ *and* $\mathfrak{t}_k \neq \mathfrak{t}_{k+2}$*, or* $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$ *and* $\mathfrak{t}_k = \mathfrak{t}_{k+2}$*, for* $1 \le k < n-1$*. Then* $S_kS_{k+1}S_kv_\mathfrak{t} = S_{k+1}S_kS_{k+1}v_\mathfrak{t}$*.*

Proof. There are again two cases to consider.

Case 1. $S_{k+1}\mathfrak{t}$ **is defined**:
Suppose first that $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$ and $\mathfrak{t}_k \neq \mathfrak{t}_{k+2}$. Then $\mathfrak{u} = S_{k+1}\mathfrak{t} \in \mathscr{T}^{ud}(\lambda)$

is well-defined. Furthermore, $\mathfrak{u}_k \neq \mathfrak{u}_{k+2}$ and $\mathfrak{u}_{k-1} \neq \mathfrak{u}_{k+1}$, so $S_kS_{k+1}S_kv_{\mathfrak{u}} = S_{k+1}S_kS_{k+1}v_{\mathfrak{u}}$ by Lemma 4.28. Now, $S_{k+1}v_{\mathfrak{u}} = a_{\mathfrak{u}}(k+1)v_{\mathfrak{u}} + b_{\mathfrak{u}}(k+1)v_{\mathfrak{t}}$ and $b_{\mathfrak{u}}(k+1) \neq 0$. Therefore

$$\begin{aligned}
S_kS_{k+1}S_kv_{\mathfrak{t}} &= \frac{1}{b_{\mathfrak{u}}(k+1)}S_kS_{k+1}S_k\Big(S_{k+1}v_{\mathfrak{u}} - a_{\mathfrak{u}}(k+1)v_{\mathfrak{u}}\Big)\\
&= \frac{1}{b_{\mathfrak{u}}(k+1)}\Big(S_k(S_{k+1}S_kS_{k+1})v_{\mathfrak{u}} - a_{\mathfrak{u}}(k+1)(S_kS_{k+1}S_k)v_{\mathfrak{u}}\Big)\\
&= \frac{1}{b_{\mathfrak{u}}(k+1)}\Big(S_k(S_kS_{k+1}S_k)v_{\mathfrak{u}} - a_{\mathfrak{u}}(k+1)(S_{k+1}S_kS_{k+1})v_{\mathfrak{u}}\Big)
\end{aligned}$$

by Lemma 4.28. Hence, using Lemma 4.24 twice,

$$\begin{aligned}
S_kS_{k+1}S_kv_{\mathfrak{t}} &= \frac{1}{b_{\mathfrak{u}}(k+1)}\Big(S_{k+1}S_kv_{\mathfrak{u}} - a_{\mathfrak{u}}(k+1)(S_{k+1}S_kS_{k+1})v_{\mathfrak{u}}\Big)\\
&= \frac{1}{b_{\mathfrak{u}}(k+1)}\Big(S_{k+1}S_k(S_{k+1}S_{k+1})v_{\mathfrak{u}} - a_{\mathfrak{u}}(k+1)(S_{k+1}S_kS_{k+1})v_{\mathfrak{u}}\Big)\\
&= \frac{1}{b_{\mathfrak{u}}(k+1)}(S_{k+1}S_kS_{k+1})\Big(S_{k+1}v_{\mathfrak{u}} - a_{\mathfrak{u}}(k+1)v_{\mathfrak{u}}\Big)\\
&= (S_{k+1}S_kS_{k+1})v_{\mathfrak{t}}
\end{aligned}$$

as required.

The case when $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$ and $\mathfrak{t}_k = \mathfrak{t}_{k+2}$ can be proved similarly.

Case 2. $S_{k+1}\mathfrak{t}$ is not defined:
This is equivalent to saying that the two nodes $\mathfrak{t}_{k+2} \ominus \mathfrak{t}_{k+1}$ and $\mathfrak{t}_{k+1} \ominus \mathfrak{t}_k$ are in the same row or in the same column. Therefore, either $\mathfrak{t}_k \subset \mathfrak{t}_{k+1} \subset \mathfrak{t}_{k+2}$ or $\mathfrak{t}_k \supset \mathfrak{t}_{k+1} \supset \mathfrak{t}_{k+2}$. Note that in either case $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$, so we have

$$E_kv_{\mathfrak{t}} = \sum_{\substack{\mathfrak{u} \overset{l}{\sim} \mathfrak{t}\\ \mathfrak{u}\neq\mathfrak{t}}} e_{\mathfrak{t}\mathfrak{u}}(k)v_{\mathfrak{u}} + e_{\mathfrak{t}\mathfrak{t}}(k)v_{\mathfrak{t}}.$$

By Lemma 4.26 and Lemma 4.25, $S_kS_{k+1}S_kE_kv_{\mathfrak{t}} = S_kS_{k+1}E_kv_{\mathfrak{t}} = E_{k+1}E_kv_{\mathfrak{t}}$ and $S_{k+1}S_kS_{k+1}E_kv_{\mathfrak{t}} = S_{k+1}E_{k+1}E_kv_{\mathfrak{t}} = E_{k+1}E_kv_{\mathfrak{t}}$.

Suppose that $\mathfrak{u} \overset{k}{\sim} \mathfrak{t}$ and $\mathfrak{u} \neq \mathfrak{t}$. Then $S_{k+1}\mathfrak{u}$ is well-defined and $\mathfrak{u}_{k-1} = \mathfrak{u}_{k+1}$—indeed, the two boxes $\mathfrak{t}_{k+2} \ominus \mathfrak{t}_{k+1}$ and $\mathfrak{t}_{k+1} \ominus \mathfrak{u}_k$ belong to different rows and columns. Hence, by Case 1, $S_{k+1}S_kS_{k+1}v_{\mathfrak{u}} = S_kS_{k+1}S_kv_{\mathfrak{u}}$. Consequently, $S_{k+1}S_kS_{k+1}e_{\mathfrak{t}\mathfrak{t}}(k)v_{\mathfrak{t}} = S_kS_{k+1}S_ke_{\mathfrak{t}\mathfrak{t}}(k)v_{\mathfrak{t}}$. Canceling the non-zero factor $e_{\mathfrak{t}\mathfrak{t}}(k)$ shows that $S_kS_{k+1}S_kv_{\mathfrak{t}} = S_kS_{k+1}v_{\mathfrak{t}}$. □

Proposition 4.30. *Suppose that $1 \leq k < n-1$ and $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$. Then $S_kS_{k+1}S_kv_{\mathfrak{t}} = S_{k+1}S_kS_{k+1}v_{\mathfrak{t}}$.*

Proof. By Lemma 4.28 and Lemma 4.29 it only remains to consider the case when $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$ and $\mathfrak{t}_k = \mathfrak{t}_{k+2}$. By Lemma 4.24, Proposition 4.25 and Proposition 4.26(a), we have

$$S_{k+1}S_kS_{k+1}E_kv_{\mathfrak{t}} = S_{k+1}S_k \cdot S_kE_{k+1}E_kv_{\mathfrak{t}} = S_{k+1}E_{k+1}E_kv_{\mathfrak{t}} = E_{k+1}E_kv_{\mathfrak{t}},$$

on the one hand. Similarly, we also have

$$S_kS_{k+1}S_kE_kv_{\mathfrak{t}} = S_kS_{k+1}E_kv_{\mathfrak{t}} = S_k \cdot S_kE_{k+1}E_kv_{\mathfrak{t}} = E_{k+1}E_kv_{\mathfrak{t}},$$

Therefore, recalling the definition of $E_kv_{\mathfrak{t}}$, we have

$$\big(S_{k+1}S_kS_{k+1} - S_kS_{k+1}S_k\big)\Big(e_{\mathfrak{t}\mathfrak{t}}(k)v_{\mathfrak{t}} + \sum_{\mathfrak{u}\overset{k}{\sim}\mathfrak{t},\mathfrak{u}\neq\mathfrak{t}} e_{\mathfrak{t}\mathfrak{u}}(k)v_{\mathfrak{u}}\Big) = 0.$$

Now, if $\mathfrak{u} \overset{k}{\sim} \mathfrak{t}$ and $\mathfrak{u} \neq \mathfrak{t}$ then $S_kS_{k+1}S_kv_\mathfrak{u} = S_{k+1}S_kS_{k+1}v_\mathfrak{u}$ by Lemma 4.25. Consequently, $S_kS_{k+1}S_kv_\mathfrak{t} = S_{k+1}S_kS_{k+1}v_\mathfrak{t}$ since $e_{\mathfrak{t}\mathfrak{t}}(k) \neq 0$. This completes the proof. □

Proof of Theorem 4.13. The results from Lemma 4.22 to Proposition 4.30 show that the action of the generators of $\mathscr{W}_{r,n}(\mathbf{u})$ on $\Delta(\lambda)$ respects all of the relations of $\mathscr{W}_{r,n}(\mathbf{u})$. Hence, $\Delta(\lambda)$ is a $\mathscr{W}_{r,n}(\mathbf{u})$–module, as we wanted to show. □

5. Irreducible representations and Theorem A

In this section we use the seminormal representations to show that the cyclotomic Wenzl algebras are always free with rank $r^n(2n-1)!!$. Before we can do this we need to recall some identities involving updown tableaux.

First, if λ is a multipartition of $n-2m$ let $f^{(n,\lambda)}$ be the number of n–updown λ–tableaux. So, in particular, $f^{(|\lambda|,\lambda)} = \#\mathscr{T}^{std}(\lambda)$ is the number of standard λ–tableaux. Sundaram [Sun86, Lemma 8.7] has given a combinatorial bijection to show that if τ is a *partition* (so $r=1$) then the number of n–updown τ–tableaux is equal to $\binom{n}{|\tau|}(n-|\tau|-1)!!f^{(|\tau|,\tau)}$. Terada [Ter01] has given a geometric version of this bijection when $|\tau|=0$ and n is even. Therefore, if λ is a multipartition then

$$\begin{aligned} f^{(n,\lambda)} &= \sum_{\substack{n_1,\dots,n_r\\ n_1+\dots+n_r=n\\ n_t-|\lambda^{(t)}|\in 2\mathbb{Z}}} \binom{n}{n_1,\dots,n_r}\prod_{t=1}^r \binom{n_t}{|\lambda^{(t)}|}(n_t-|\lambda^{(t)}|-1)!!f^{(|\lambda^{(t)}|,\lambda^{(t)})}\\ &= \sum_{\substack{n_1,\dots,n_r\\ n_1+\dots+n_r=n\\ n_t-|\lambda^{(t)}|\in 2\mathbb{Z}}} n!\prod_{t=1}^r \frac{(n_t-|\lambda^{(t)}|-1)!!f^{(|\lambda^{(t)}|,\lambda^{(t)})}}{(n_t-|\lambda^{(t)}|)!|\lambda^{(t)}|!} \end{aligned}$$

Suppose for the moment that $r=1$ and note that $f^{(|\tau|,\tau)}$ is equal to the number of standard τ–tableaux. Then it is well–known that $\sum_{\tau\vdash k} {f^{(k,\tau)}}^2 = k!$.

Lemma 5.1. *Suppose that $n\geq 1$ and $r\geq 1$. Then*

$$\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{\lambda\vdash n-2m} {f^{(n,\lambda)}}^2 = r^n(2n-1)!!.$$

Proof. If $r\geq 1$ then using the formulas before the Lemma, the left hand side is equal to

$$\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{\substack{k_1,\dots,k_r\\ k_1+\dots+k_r=n-2m}}\sum_{\substack{n_1,\dots,n_r\\ n_1+\dots+n_r=n\\ n_t-k_t\in 2\mathbb{Z}}}\sum_{\substack{n'_1,\dots,n'_r\\ n'_1+\dots+n'_r=n\\ n'_t-k_t\in 2\mathbb{Z}}} (n!)^2\prod_{t=1}^r\frac{(n_t-k_t-1)!!(n'_t-k_t-1)!!}{k_t!(n_t-k_t)!(n'_t-k_t)!}.$$

Changing the sums over n_t and n'_t, to the sums over $a_t=\frac{n_t-k_t}{2}$ and $a'_t=\frac{n'_t-k_t}{2}$, this becomes

$$(n!)^2\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\Big(\sum_{\substack{k_1,\dots,k_r\\ k_1+\dots+k_r=n-2m}}\prod_{t=1}^r\frac{1}{k_t!}\Big)\Big(\sum_{\substack{a_1,\dots,a_r\\ a_1+\dots+a_r=m}}\prod_{t=1}^r\frac{2^{-a_t}}{a_t!}\Big)\Big(\sum_{\substack{a'_1,\dots,a'_r\\ a'_1+\dots+a'_r=m}}\prod_{t=1}^r\frac{2^{-a'_t}}{a'_t!}\Big).$$

Thus,

$$\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{\lambda\vdash n-2m} {f^{(n,\lambda)}}^2 = r^n\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(n!)^2 2^{-2m}}{(m!)^2(n-2m)!}.$$

To complete the proof, notice that the sum on the right hand side does not depend on r, so we can set $r=1$ to deduce the result. $\square$

A representation theoretic proof of this result is given in [RY04] where it is obtained as a consequence of the branching rules for the cyclotomic Brauer algebra. The cell modules of the cyclotomic Brauer algebras are indexed by the multipartitions of $n-2m$, for $0\le m\le\lfloor\frac{n}{2}\rfloor$. The branching rule [RY04, Theorem 6.1] shows that the dimension of the cell module indexed by λ is $f^{(n,\lambda)}$. On the other hand, the cellular basis of the cyclotomic Brauer algebras constructed in [RY04, Theorem 5.11] contains $r^n(2n-1)!!$ elements. Combining these two facts proves the result.

Given two multipartitions λ and μ such that μ is obtained by adding a box to λ we write $\lambda\to\mu$, or $\mu\leftarrow\lambda$.

Theorem 5.2. *Suppose that R is a field with $\operatorname{char} R\ge 2n$ and the root conditions (Assumption 4.12) hold in R. Assume that the parameters $u_1,\dots,u_r$ are generic for $\mathscr{W}_{r,n}(\mathbf{u})$ and that Ω is $\mathbf{u}$–admissible. Then:*

a) *Suppose $n>1$. There is a $\mathscr{W}_{r,n-1}(\mathbf{u})$-module isomorphism*
$$\Delta(\lambda)\downarrow=\bigoplus_{\substack{\mu\\ \mu\to\lambda}}\Delta(\mu)\quad\bigoplus\quad\bigoplus_{\substack{\nu\\ \lambda\to\nu}}\Delta(\nu).$$
where $\Delta(\lambda)\downarrow$ is $\Delta(\lambda)$ considered as a $\mathscr{W}_{r,n-1}(\mathbf{u})$-module.

b) *The seminormal representation $\Delta(\lambda)$ is an irreducible $\mathscr{W}_{r,n}(\mathbf{u})$–module for each multipartition λ of $n-2m$, where $0\le m\le\lfloor\frac{n}{2}\rfloor$.*

c) *The set $\{\,\Delta(\lambda)\mid\lambda\vdash n-2m, 0\le m\le\lfloor\frac{n}{2}\rfloor\,\}$ is a complete set of irreducible $\mathscr{W}_{r,n}(\mathbf{u})$–modules.*

d) *$\mathscr{W}_{r,n}(\mathbf{u})$ is a semisimple R–algebra of dimension $r^n(2n-1)!!$.*

Proof. Part (a) follows if we define $\Delta(\mu)$ to be the vector subspace spanned by $v_\mathfrak{u}$ with $\mathfrak{u}\in\mathscr{T}^{ud}_n(\lambda)$ and $\mathfrak{u}_{n-1}=\mu$.

Let $\mathscr{X}=\langle X_1,\dots,X_n\rangle$. Since $X_kv_\mathfrak{t}=c_\mathfrak{t}(k)v_\mathfrak{t}$, for all $\mathfrak{t}\in\mathscr{T}^{ud}_n(\lambda)$ and $1\le k\le n$, the seminormal representation $\Delta(\lambda)=\bigoplus_{\mathfrak{t}\in\mathscr{T}^{ud}_n(\lambda)}Rv_\mathfrak{t}$ decomposes into a direct sum of one dimensional submodules as an $\mathscr{X}$–module. Further, by Lemma 4.4(a), this decomposition is multiplicity free. In particular, $\Delta(\lambda)\cong\Delta(\mu)$ if and only if $\lambda=\mu$. Further, if M is a $\mathscr{W}_{r,n}(\mathbf{u})$-submodule of $\Delta(\lambda)$ then M is spanned by some subset of $\{\,v_\mathfrak{t}\mid\mathfrak{t}\in\mathscr{T}^{ud}_n(\lambda)\,\}$.

To prove (b) we now argue by induction on n. If $n=1$ then $\Delta(\lambda)$ is one dimensional and hence irreducible, for all λ. Suppose now that $n>1$ and let $M\subset\Delta(\lambda)$ be a non–zero $\mathscr{W}_{r,n}(\mathbf{u})$–submodule of $\Delta(\lambda)$. By the remarks in the last paragraph, M is spanned by a subset of $\{\,v_\mathfrak{t}\mid\mathfrak{t}\in\mathscr{T}^{ud}_n(\lambda)\,\}$. Therefore, if we consider M as a $\mathscr{W}_{r,n-1}(\mathbf{u})$–module then $M\supset\Delta(\mu)$, for some multipartition μ which is obtained by adding or removing a node from λ.

Case 1. $|\lambda|=n$:
Since $|\lambda|=n$, The multipartition μ is obtained from λ by removing a node. If $\lambda=((0),\dots,(0),(a^b),(0),\dots,(0))$ then $\Delta(\lambda)\downarrow$ is irreducible as a $\mathscr{W}_{r,n-1}(\mathbf{u})$–module, so there is nothing to prove. Suppose then that λ is not of this form and that ν is a different multipartition which is obtained from λ by removing a node. Let $\mathfrak{t}\in\mathscr{T}^{ud}_n(\lambda)$ be an updown tableau such that $\mathfrak{t}_{n-1}=\mu$ and $\mu\setminus\mathfrak{t}_{n-2}=\lambda\setminus\nu$. So $v_\mathfrak{t}\in\Delta(\mu)\subset M$ and $(S_{n-1}\mathfrak{t})_{n-1}=\nu$. Now,
$$S_{n-1}v_\mathfrak{t}=a_\mathfrak{t}(n-1)v_\mathfrak{t}+b_\mathfrak{t}(n-1)v_{S_{n-1}\mathfrak{t}}\in M,$$
and $b_\mathfrak{t}(n-1)\ne 0$ since $\lambda\backslash\mu$ and $\lambda\backslash\nu$ cannot be in the same row or in the same column. Consequently, $v_{S_{n-1}\mathfrak{t}}\in M$. This implies that $\Delta(\nu)\subset M$ since

$(S_{n-1}\mathfrak{t})_{n-1} = \nu$. Therefore, $\sum_{\nu\to\lambda}\Delta(\nu)\subset M$, so $M=\Delta(\lambda)$ by part(a). Hence, $\Delta(\lambda)$ is irreducible as required.

Case 2. $|\lambda|<n$:
Since $|\lambda|<n$, $\mathscr{T}^{ud}_{n-2}(\lambda)$ is non–empty so we fix $\mathfrak{u}\in\mathscr{T}^{ud}_{n-2}(\lambda)$. Let $\mathfrak{t}=(\mathfrak{u}_1,\dots,\mathfrak{u}_{n-2},\mu,\lambda)$, then $\mathfrak{t}\in\mathscr{T}^{ud}_n(\lambda)$ and $v_\mathfrak{t}\in\Delta(\mu)\subset M$. Then

$$E_{n-1}v_\mathfrak{t}=\sum_{\mathfrak{w}\overset{n-1}{\sim}\mathfrak{t}} e_{\mathfrak{t}\mathfrak{w}}(n-1)v_\mathfrak{w}\in M.$$

As $e_{\mathfrak{t}\mathfrak{w}}(n-1)\neq 0$ whenever $\mathfrak{w}\overset{n-1}{\sim}\mathfrak{t}$, we have $v_\mathfrak{w}\in M$ for each term in this sum. If $\nu\leftarrow\lambda$ or $\nu\to\lambda$ then $\mathfrak{w}=(\mathfrak{u}_1,\dots,\mathfrak{u}_{n-2},\nu,\lambda)\overset{n-1}{\sim}\mathfrak{t}$, so $\Delta(\nu)\subset M$. Hence, $M=\Delta(\lambda)$ and $\Delta(\lambda)$ is irreducible as claimed. This completes the proof of (b).

Finally, we prove (c) and (d). We have already seen that the seminormal representations are pairwise non–isomorphic, so it remains to show that every irreducible is isomorphic to $\Delta(\lambda)$ for some λ. Let $\operatorname{Rad}\mathscr{W}_{r,n}(\mathbf{u})$ be the Jacobson radical of $\mathscr{W}_{r,n}(\mathbf{u})$. Then

$$\dim_R\mathscr{W}_{r,n}(\mathbf{u})\ge\dim_R(\mathscr{W}_{r,n}(\mathbf{u})/\operatorname{Rad}\mathscr{W}_{r,n}(\mathbf{u}))\ge\sum_{m=0}^{\lfloor n/2\rfloor}\sum_{\lambda\vdash n-2m}\big(\dim_R\Delta(\lambda)\big)^2.$$

By construction, $\dim\Delta(\lambda)=\#\mathscr{T}^{ud}_n(\lambda)=f^{(n,\lambda)}$. So using Lemma 5.1, and then Proposition 2.16, we have

$$\dim_R\mathscr{W}_{r,n}(\mathbf{u})\ge r^n(2n-1)!!\ge\dim_R\mathscr{W}_{r,n}(\mathbf{u}).$$

Therefore, $\operatorname{Rad}\mathscr{W}_{r,n}(\mathbf{u})=0$, which forces $\dim_R\mathscr{W}_{r,n}(\mathbf{u})=r^n(2n-1)!!$. Now, parts (c) and (d) both follow from the Wedderburn-Artin Theorem. □

Before establishing a strong version of Theorem A, we show that the Root conditions (Assumption 4.12) can be satisfied when $R=\mathbb{R}$.

Lemma 5.3. *Suppose that $R=\mathbb{R}$ and we choose $u_i\in R$ in such a way that*
a) $|u_1|>\dots>|u_r|\ge n$ and $|u_i|-|u_{i+1}|\ge 2n$,
b) $u_i<0$ if i is even and $u_i>0$ if i is odd.
Suppose that $\mathfrak{t}\in\mathscr{T}^{ud}_n(\lambda)$ and $1\le k<n$. Then $|a_\mathfrak{t}(k)|\le 1$, if $\mathfrak{t}_{k-1}\neq\mathfrak{t}_{k+1}$, and $e_{\mathfrak{t}\mathfrak{t}}(k)>0$, if $\mathfrak{t}_{k-1}=\mathfrak{t}_{k+1}$. In particular, the Root Condition (4.12) holds if we choose positive square roots $\sqrt{b_\mathfrak{t}(k)}>0$ and $\sqrt{e_{\mathfrak{t}\mathfrak{t}}(k)}>0$.

Proof. We start with the case $\mathfrak{t}_{k-1}\neq\mathfrak{t}_{k+1}$. Let $\alpha=\mathfrak{t}_k\ominus\mathfrak{t}_{k-1}$ and $\beta=\mathfrak{t}_{k+1}\ominus\mathfrak{t}_k$. Note that $c(\alpha)+c(\beta)\neq 0$. Write $\alpha=(i,j,t)$ and $\beta=(i',j',t')$. If $t=t'$ and both nodes are addable, or both nodes are removable, then $\alpha\neq\beta$. Thus, $c(\beta)-c(\alpha)$ is a nonzero integer and $|a_\mathfrak{t}(k)|\le 1$. If $t=t'$ and only one of the nodes is addable (and the other is removable), then

$$|c(\alpha)-c(\beta)|=|2u_t+(j-i)+(j'-i')|\ge 2|u_t|-2(n-1)\ge 1.$$

A similar argument shows that $|a_\mathfrak{t}(k)|\le 1$ when $t\neq t'$.

Next we consider the case $\mathfrak{t}_{k-1}=\mathfrak{t}_{k+1}$. Let $\alpha=\mathfrak{t}_k\ominus\mathfrak{t}_{k-1}$ and $\lambda=\mathfrak{t}_{k-1}$. Write $\alpha=(i,j,t)$. By (4.8) and because $R=\mathbb{R}$, we have

$$e_{\mathfrak{t}\mathfrak{t}}(k)=\big(2c(\alpha)-(-1)^r\big)\prod_\beta\frac{c(\alpha)+c(\beta)}{c(\alpha)-c(\beta)},$$

where β runs over all of the addable and removable nodes of λ with $\beta\neq\alpha$.

Suppose that t is even. First we show that

$$\prod_{\beta\notin\lambda^{(t)}}\frac{c(\alpha)+c(\beta)}{c(\alpha)-c(\beta)}<0.$$

Consider the contents of all of the addable and removable nodes of $\lambda^{(t')}$, where $t' \neq t$. If t' is even then there are l positive contents $|u_{t'}| + d_j$ with $|d_j| < n$, for $1 \le j \le l$, and $l+1$ negative contents $-|u_{t'}| - c_i$ with $|c_i| < n$, for $1 \le i \le l+1$. Let $\varepsilon_{t'}$ be the sign of the product of $\frac{c(\alpha)+c(\beta)}{c(\alpha)-c(\beta)}$ over all addable and removable nodes!β of $\lambda^{(t')}$. Our aim is to show that

$$\prod_{t'\neq t} \varepsilon_{t'} = -1.$$

By our assumptions, $\varepsilon_{t'}$ is equal to the sign of

$$\frac{(-|u_t| + |u_{t'}|)^l}{(-|u_t| - |u_{t'}|)^l} \frac{(-|u_t| - |u_{t'}|)^{l+1}}{(-|u_t| + |u_{t'}|)^{l+1}} = \frac{|u_t| + |u_{t'}|}{|u_t| - |u_{t'}|}.$$

Thus, $\varepsilon_{t'} < 0$ if and only if $t' < t$. If t' is odd then there are $l+1$ positive contents $|u_{t'}| + c_i$ with $|c_i| < n$, for $1 \le i \le l+1$, and l negative contents $-|u_{t'}| - d_j$ with $|d_j| < n$, for $1 \le j \le l$. Then, by the same argument, $\varepsilon_{t'} < 0$ if and only if $t' < t$ again. Thus

$$\prod_{t'\neq t} \varepsilon_{t'} = (-1)^{t-1} = -1.$$

Let $-|u_t| - c_i$, for $1 \le i \le l+1$, be the contents of the addable nodes of $\lambda^{(t)}$ and let $|u_t| + d_j$, for $1 \le j \le l$, be the contents of the removable nodes of $\lambda^{(t)}$. We may assume that

$$c_1 > d_1 > \cdots > c_l > d_l > c_{l+1}.$$

Let ε_t be the sign of the product of $\frac{c(\alpha)+c(\beta)}{c(\alpha)-c(\beta)}$, where β runs over all of the addable and removable nodes of $\lambda^{(t)}$ such that $\beta \neq \alpha$.

If $c(\alpha) = -|u_t| - c_i$, for some i, then ε_t is equal to the sign of

$$\prod_{k\neq i} \frac{-2|u_t| - c_i - c_k}{c_k - c_i} \prod_{k=1}^{l} \frac{d_k - c_i}{-2|u_t| - c_i - d_k},$$

so $\varepsilon_t = \frac{(-1)^l}{(-1)^{l+1-i}} \frac{(-1)^{l-i+1}}{(-1)^l} = 1$. As $2c(\alpha) - (-1)^r = -2|u_t| - 2c_i \pm 1 < 0$ and

$$\prod_{1\le t'\le r} \varepsilon_{t'} = -1,$$

we have $e_{\mathfrak{t}\mathfrak{t}}(k) > 0$.

If $c(\alpha) = |u_t| + d_j$, for some j, then ε_t is equal to the sign of

$$\prod_{k=1}^{l+1} \frac{d_j - c_k}{2|u_t| + d_j + c_k} \prod_{k\neq j}^{l} \frac{2|u_t| + d_j + d_k}{d_j - d_k},$$

so $\varepsilon_t = \frac{(-1)^j}{(-1)^{j-1}} = -1$. As $2c(\alpha) - (-1)^r = 2|u_t| + 2d_j \pm 1 > 0$ and

$$\prod_{1\le t'\le r} \varepsilon_{t'} = -1,$$

we have $e_{\mathfrak{t}\mathfrak{t}}(k) > 0$ again.

The case when t is odd is handled similarly. In this case, we have

$$\prod_{\beta\notin\lambda^{(t)}} \frac{c(\alpha) + c(\beta)}{c(\alpha) - c(\beta)} > 0,$$

because its sign is equal to $(-1)^{t-1} = 1$. Let $|u_t| + c_i$, for $1 \le i \le l+1$, be the contents of the addable nodes of $\lambda^{(t)}$ and let $-|u_t| - d_j$, for $1 \le j \le l$, be the contents of the removable nodes of $\lambda^{(t)}$ such that

$$c_1 > d_1 > \cdots > c_l > d_l > c_{l+1}.$$

If $c(\alpha) = |u_t| + c_i$, for some i, then ε_t is equal to the sign of

$$\prod_{k\neq i} \frac{2|u_t| + c_i + c_k}{c_i - c_k} \prod_{k=1}^{l} \frac{c_i - d_k}{2|u_t| + c_i + d_k},$$

so $\varepsilon_t = \frac{(-1)^{i-1}}{(-1)^{i-1}} = 1$. As $2c(\alpha) - (-1)^r > 0$ we have $e_{\mathfrak{t}\mathfrak{t}}(k) > 0$.

If $c(\alpha) = -|u_t| - d_j$, for some j, then ε_t is equal to the sign of

$$\prod_{k=1}^{l+1} \frac{c_k - d_j}{-2|u_t| - d_j - c_k} \prod_{k\neq j}^{l} \frac{-2|u_t| - d_j - d_k}{d_k - d_j},$$

so $\varepsilon_t = \frac{(-1)^{l-j+1}}{(-1)^{l+1}} \frac{(-1)^{l-1}}{(-1)^{l-j}} = -1$. As $2c(\alpha) - (-1)^r < 0$ we have $e_{\mathfrak{t}\mathfrak{t}}(k) > 0$ again. $\square$

We can now prove a stronger version of Theorem A.

Theorem 5.4. *Suppose that R is a commutative ring of in which 2 is invertible and that Ω is $\mathbf{u}$–admissible. Then $\mathscr{W}_{r,n}(\mathbf{u})$ is free as an R–module with basis the set of r–regular monomials. Consequently, $\mathscr{W}_{r,n}(\mathbf{u})$ is free of rank $r^n(2n-1)!!$.*

Proof. Recall that if R is a ring in which 2 is invertible then $\mathscr{W}_{r,n}(\mathbf{u})$ is spanned by the set of r-regular monomials by Proposition 2.16. For convenience, if S is a ring and $\mathbf{u}_s \in S^r$ then we let $\mathscr{W}_S(\mathbf{u}_S)$ be the cyclotomic Wenzl algebra defined over S with parameters $\mathbf{u}_S$.

First, we consider the special case when $R = \mathcal{Z}$, where $\mathcal{Z} = \mathbb{Z}[\frac{1}{2}, \dot{u}_1, \dots, \dot{u}_r]$ and the $\dot{u}_i$ are indeterminates over $\mathbb{Z}$. Let $\dot{\mathbf{u}} = (\dot{u}_1, \dots, \dot{u}_r)$, define $\dot{\Omega}$ in accordance with Definition 3.5 and consider the cyclotomic Wenzl algebra $\mathscr{W}_{\mathcal{Z}}(\dot{\mathbf{u}})$. As $\mathbb{R}$ is not finitely generated over $\mathbb{Q}$ we can find r algebraically independent transcendental real numbers $u'_1, \dots, u'_r \in \mathbb{R}$ which satisfy the hypotheses of Lemma 5.3. Let $\mathcal{Z}' = \mathbb{Z}[\frac{1}{2}, u'_1, \dots, u'_r]$ and let $\theta : \mathcal{Z} \longrightarrow \mathcal{Z}'$ be the $\mathcal{Z}$–linear map determined by $\theta(\dot{u}_i) = u'_i$, for $1 \le i \le r$. Then θ is a ring isomorphism. Let $\mathbf{u}' = (u'_1, \dots, u'_r)$ and $\Omega' = \{\, \theta(\dot{\omega}_a) \mid a \ge 0 \,\}$. Then Ω' is $\mathbf{u}'$–admissible and θ induces an isomorphism of $\mathcal{Z}$–algebras $\mathscr{W}_{\mathcal{Z}}(\dot{\mathbf{u}}) \cong \mathscr{W}_{\mathcal{Z}'}(\mathbf{u}')$, where the inverse map is the homomorphism induced by $\theta^{-1} : \mathcal{Z}' \longrightarrow \mathcal{Z}$.

Now, by Lemma 5.3 and Theorem 5.2(d), $\mathscr{W}_{\mathbb{R}}(\mathbf{u}')$ is an $\mathbb{R}$–algebra of dimension $r^n(2n-1)!!$. Hence the set of r–regular monomials is an $\mathbb{R}$–basis of $\mathscr{W}_{\mathbb{R}}(\mathbf{u}')$ since there are $r^n(2n-1)!!$ r-regular monomials. In particular, the set of r-regular monomials is linearly independent over $\mathbb{R}$, and hence linearly independent over $\mathcal{Z}'$. Therefore, $\mathscr{W}_{\mathcal{Z}'}(\mathbf{u}')$ is free as a $\mathcal{Z}'$-module of rank $r^n(2n-1)!!$. Hence, $\mathscr{W}_{\mathcal{Z}}(\dot{\mathbf{u}})$ is free as a $\mathcal{Z}$-module of rank $r^n(2n-1)!!$.

Now suppose that R is an arbitrary commutative ring (in which 2 is invertible). Then we can consider R as a $\mathcal{Z}$–algebra by letting $\dot{u}_i$ act on R as multiplication by u_i, for $1 \le i \le r$. Since $\mathscr{W}_{\mathcal{Z}}(\dot{\mathbf{u}})$ is $\mathcal{Z}$–free, the R–algebra $\mathscr{W}_{\mathcal{Z}}(\dot{\mathbf{u}}) \otimes_{\mathcal{Z}} R$ is free as an R–module of rank $r^n(2n-1)!!$. As the generators of $\mathscr{W}_{\mathcal{Z}}(\dot{\mathbf{u}}) \otimes_{\mathcal{Z}} R$ satisfy the relations of $\mathscr{W}_{r,n}(\mathbf{u}) = \mathscr{W}_R(\mathbf{u})$ we have a surjective homomorphism $\mathscr{W}_{r,n}(\mathbf{u}) \longrightarrow \mathscr{W}_{\mathcal{Z}}(\dot{\mathbf{u}}) \otimes_{\mathcal{Z}} R$. By Proposition 2.16 this map must be an isomorphism, so we are done. $\square$

As an easy application of the Theorem we obtain the following useful fact which we will use many times below without mention.

Proposition 5.5. *Suppose that R is a commutative ring of in which 2 is invertible and that Ω is $\mathbf{u}$–admissible.*

- *a) For $1 \le m \le n$, let $\mathscr{W}'_{r,m}(\mathbf{u})$ be the subalgebra of $\mathscr{W}_{r,n}(\mathbf{u})$ generated by $\{\, S_i, E_i, X_j \mid 1 \le i < m \text{ and } 1 \le j \le m \,\}$. Then $\mathscr{W}'_{r,m}(\mathbf{u}) \cong \mathscr{W}_{r,m}(\mathbf{u})$.*
- *b) The Brauer algebra $\mathscr{B}_n(\omega_0)$ is isomorphic to the subalgebra of $\mathscr{W}_{r,n}(\mathbf{u})$ generated by $\{\, S_i, E_i \mid 1 \le i < n \,\}$.*

6. The degenerate Hecke algebras of type $G(r,1,n)$

Suppose R is a commutative ring and let $\mathbf{u}\in R^r$. Recall from section 2 that $\mathscr{H}_{r,n}(\mathbf{u})$ is the degenerate Hecke algebra with parameters $\mathbf{u}$. In this section we give several results from the representation theory of $\mathscr{H}_{r,n}(\mathbf{u})$ which we will need in our study of the cyclotomic Wenzl algebras. As the proofs of these results are very similar to (and easier than) the proofs of the corresponding results for the Ariki–Koike algebras we are very brief with the details.

The following result is proved by Kleshchev [Kle05]. We use the seminormal representations of $\mathscr{W}_{r,n}(\mathbf{u})$ to give another proof.

Let $\Lambda_r^+(n)$ be the set of r–multipartitions of n. We consider $\Lambda_r^+(n)$ as a partially ordered set under dominance $\trianglerighteq$, where $\lambda\trianglerighteq\mu$ if

$$\sum_{t=s+1}^{r}|\lambda^{(t)}|+\sum_{j=1}^{k}\lambda_k^{(t)}\geq\sum_{t=s+1}^{r}|\mu^{(t)}|+\sum_{j=1}^{k}\mu_k^{(t)},$$

for $1\le s\le r$ and all $k\ge 0$. If $\lambda\trianglerighteq\mu$ and $\lambda\ne\mu$ we sometimes write $\lambda\triangleright\mu$.

Theorem 6.1. *The degenerate Hecke algebra $\mathscr{H}_{r,n}(\mathbf{u})$ is free as an R–module of rank $r^n n!$.*

Proof. It is not difficult to see that for any ring R set

$$\{\,Y_1^{k_1}Y_2^{k_2}\cdots Y_n^{k_n}T_w\mid 0\le k_i\le r-1, w\in\mathfrak{S}_n\,\}$$

spans $\mathscr{H}_{r,n}(\mathbf{u})$ as an R-module. So we need to prove that these elements are linearly independent.

We adopt the notation from the proof of Theorem 5.4. As in the proof of that result, we first consider the case when $R=\mathcal{Z}$, where $\mathcal{Z}=\mathbb{Z}[\frac12,\dot u_1,\dots,\dot u_r]$, and we choose r algebraically independent transcendental real numbers $u_1',\dots,u_r'$ which satisfy the hypotheses of Lemma 5.3. Let $\mathcal{Z}'=\mathbb{Z}[\frac12,u_1',\dots,u_r']$. Then $\mathcal{Z}\cong\mathcal{Z}'\hookrightarrow\mathbb{R}$ and we can ask whether the degenerate Hecke algebra $\mathscr{H}_{\mathbb{R}}(\mathbf{u}')$, defined over $\mathbb{R}$ and with parameters $\mathbf{u}'=(u_1',\dots,u_r')$, acts on the seminormal representations of $\mathscr{W}_{\mathbb{R}}(\mathbf{u}')$. By definition, if $\lambda\in\Lambda_r^+(n)$ then $E_i\Delta(\lambda)=0$, for $1\le i<n$. Therefore, over $\mathbb{R}$, $\Delta(\lambda)$ can be considered as an $\mathscr{H}_{\mathbb{R}}(\mathbf{u}')$–module by Corollary 2.17. Hence, as in the proof of Theorem 5.4,

$$\dim_{\mathbb{R}}\mathscr{H}_{\mathbb{R}}(\mathbf{u}')\geq\sum_{\lambda\in\Lambda_r^+(n)}(\dim_{\mathbb{R}}\Delta(\lambda))^2=r^n n!.$$

Consequently, by the opening paragraph of the proof, this set is a basis of $\mathscr{H}_{\mathbb{R}}(\mathbf{u}')$. As in the proof of Theorem 5.4 it follows that $\mathscr{H}_{\mathcal{Z}}(\dot{\mathbf{u}})$ is free as a $\mathcal{Z}$–module of rank $r^n n!$. The result for a general ring R now follows by a specialization argument. $\square$

We remark the definition seminormal representations of $\mathscr{W}_{r,n}(\mathbf{u})$ required that R satisfy assumption (4.12). It is not hard to modify the definition of the seminormal representations of $\mathscr{H}_{r,n}(\mathbf{u})$ so that they do not require the taking of square roots and so work over an arbitrary field (cf. [AK94]), which gives simplification of the argument above.

Definition 6.2 (Graham and Lehrer [GL96])**.** Let R be a commutative ring and A an R–algebra. Fix a partially ordered set $\Lambda=(\Lambda,\trianglerighteq)$ and for each $\lambda\in\Lambda$ let $T(\lambda)$ be a finite set. Finally, fix $C_{\mathfrak{st}}^\lambda\in A$ for all $\lambda\in\Lambda$ and $\mathfrak{s},\mathfrak{t}\in T(\lambda)$.

Then the triple (Λ,T,C) is a cell datum for A if:

a) $\{\,C_{\mathfrak{st}}^\lambda\mid\lambda\in\Lambda\text{ and }\mathfrak{s},\mathfrak{t}\in T(\lambda)\,\}$ is an R–basis for A;

b) the R–linear map $*\colon A\longrightarrow A$ determined by $(C_{\mathfrak{st}}^\lambda)^*=C_{\mathfrak{ts}}^\lambda$, for all $\lambda\in\Lambda$ and all $\mathfrak{s},\mathfrak{t}\in T(\lambda)$ is an anti–isomorphism of A;

c) for all $\lambda \in \Lambda$, $\mathfrak{s} \in T(\lambda)$ and $a \in A$ there exist scalars $r_{\mathfrak{s}\mathfrak{u}}(a) \in R$ such that

$$aC^{\lambda}_{\mathfrak{s}\mathfrak{t}} = \sum_{\mathfrak{u}\in T(\lambda)} r_{\mathfrak{s}\mathfrak{u}}(a)C^{\lambda}_{\mathfrak{u}\mathfrak{t}} \pmod{A^{\rhd\lambda}},$$

where $A^{\rhd\lambda} = R\text{–span}\{\, C^{\mu}_{\mathfrak{u}\mathfrak{v}} \mid \mu \rhd \lambda \text{ and } \mathfrak{u}, \mathfrak{v} \in T(\mu)\,\}$.

An algebra A is a cellular algebra if it has a cell datum and in this case we call $\{\, C^{\lambda}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in T(\lambda), \lambda \in \Lambda\,\}$ a cellular basis of A.

To show that $\mathscr{H}_{r,n}(\mathbf{u})$ is a cellular algebra we modify the construction of the Murphy basis of the Ariki–Koike algebras; see [DJM99]. For any multipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(r)})$ we define $u_\lambda = u_{a_1,1}u_{a_2,2}\cdots u_{a_{r-1},r-1}$, where $u_{a,i} = (Y_1 - u_{i+1})(Y_2 - u_{i+1})\cdots(Y_a - u_{i+1})$ and $a_i = \sum_{j=1}^{i} |\lambda^{(j)}|$, $1 \le i \le r-1$. Let $\mathfrak{S}_\lambda$ be the Young subgroup $\mathfrak{S}_{\lambda^{(1)}} \times \mathfrak{S}_{\lambda^{(2)}} \times \cdots \times \mathfrak{S}_{\lambda^{(r)}}$ of $\mathfrak{S}_n$. Let $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w$ and define

$$m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})^{-1}}u_\lambda x_\lambda T_{d(\mathfrak{t})} \quad \in \mathscr{H}_{r,n}(\mathbf{u}),$$

where $\mathfrak{s}, \mathfrak{t}$ are standard λ-tableaux.

Theorem 6.3. *The set* $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}(\lambda)$ *and* $\lambda \in \Lambda^+_r(n)\,\}$ *is a cellular basis of* $\mathscr{H}_{r,n}(\mathbf{u})$.

Proof. The proof of this result is similar to, but much easier than, the corresponding result for the cyclotomic Hecke algebras. See [DJM99] for details. □

We next give a formula for the Gram determinant of the cell modules of $\mathscr{H}_{r,n}(\mathbf{u})$. This requires some definitions.

Definition 6.4. The parameters $\mathbf{u} = (u_1, \dots, u_r)$ are **generic** for $\mathscr{H}_{r,n}(\mathbf{u})$ if whenever there exists $d \in \mathbb{Z}$ such that $u_i - u_j = d \cdot 1_R$ then $|d| \ge n$.

Recall that if λ is a multipartition of n then there is a natural bijection between the set of n–updown λ–tableaux in the sense of [DJM99]. Accordingly, we set $\mathscr{T}^{std}(\lambda) = \mathscr{T}^{ud}_n(\lambda)$ and refer to the elements of $\mathscr{T}^{std}(\lambda)$ as standard λ–tableaux. The following Lemma is well-known (cf. [JM00, Lemma 3.12]), and is easily verified by induction on n.

Lemma 6.5. *Suppose that the parameters* $\mathbf{u}$ *are generic for* $\mathscr{H}_{r,n}(\mathbf{u})$ *and that* R *is a field with* $\operatorname{char} R > n$. *Let* λ *and* μ *be multipartitions of* n *and suppose that* $\mathfrak{s} \in \mathscr{T}^{std}_n(\lambda)$ *and* $\mathfrak{t} \in \mathscr{T}^{std}_n(\mu)$. *Then* $\mathfrak{s} = \mathfrak{t}$ *if and only if* $c_\mathfrak{s}(k) = c_\mathfrak{t}(k)$, *for* $k = 1, \dots, n$.

As in the definition of a cellular basis, if $\lambda \in \Lambda^+_r(n)$ then we let $\mathscr{H}^{\rhd\lambda}_{r,n}$ be the free R–submodule $\mathscr{H}_{r,n}$ with basis $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}(\mu) \text{ for } \mu \rhd \lambda\,\}$. It follows directly from Definition 6.2(c) that $\mathscr{H}^{\rhd\lambda}_{r,n}$ is a two–sided ideal of $\mathscr{H}_{r,n}$.

Lemma 6.6. *Suppose that* λ *is a multipartition of* n *and that* $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}_n(\lambda)$. *Then*

$$Y_k m_{\mathfrak{s}\mathfrak{t}} = c_\mathfrak{s}(k)m_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{u}\in\mathscr{T}^{std}_n(\lambda)\\ \mathfrak{u}\rhd\mathfrak{s}}} r_{\mathfrak{u}\mathfrak{t}}Y_{\mathfrak{u}\mathfrak{t}} \pmod{\mathscr{H}^{\rhd\lambda}_{r,n}},$$

for some $r_{\mathfrak{u}\mathfrak{t}} \in R$.

Proof. If $r = 1$ then this is a result of Murphy's [Mur83]. The general case can be deduced from this following the argument of [JM00, Prop. 3.7]. □

We can now follow the arguments of [Mat04] to construct a "seminormal" basis of $\mathscr{H}_{r,n}$.

Definition 6.7. Suppose that $\lambda \in \Lambda^+_r(n)$.

a) For each $\mathfrak{t} \in \mathscr{T}^{std}(\lambda)$ let

$$F_{\mathfrak{t}} = \prod_{k=1}^{n} \prod_{\substack{\mu \in \Lambda_r^+(n) \\ \mathfrak{u} \in \mathscr{T}^{std}(\mu) \\ c_{\mathfrak{u}}(k) \neq c_{\mathfrak{t}}(k)}} \frac{Y_k - c_{\mathfrak{u}}(k)}{c_{\mathfrak{t}}(k) - c_{\mathfrak{u}}(k)}.$$

b) If $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}(\lambda)$ then let $f_{\mathfrak{s}\mathfrak{t}} = F_{\mathfrak{s}} m_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{t}}$.

Using the last two results and the definitions it is not hard to show that if $\mathfrak{s}, \mathfrak{t}$ and $\mathfrak{u}$ are standard tableaux then $f_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{u}} = \delta_{\mathfrak{t}\mathfrak{u}} f_{\mathfrak{s}\mathfrak{t}}$; see, for example, [Mat99, Prop. 3.35]. Hence, from Theorem 6.3 and Lemma 6.6 we obtain the following.

Proposition 6.8. *Suppose that R is a field with $\operatorname{char} R > n$ and that $\mathbf{u}$ is generic for $\mathscr{H}_{r,n}(\mathbf{u})$. Then $\{ f_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}(\lambda), \lambda \in \Lambda_r^+(n) \}$ is a basis of $\mathscr{H}_{r,n}(\mathbf{u})$. Moreover, for each standard tableau $\mathfrak{t}$ there exists a scalar $\gamma_{\mathfrak{t}} \in R$ such that*

$$f_{\mathfrak{s}\mathfrak{t}} f_{\mathfrak{u}\mathfrak{v}} = \delta_{\mathfrak{t}\mathfrak{u}} \gamma_{\mathfrak{t}} f_{\mathfrak{s}\mathfrak{v}},$$

where $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}(\lambda)$, $\mathfrak{u}, \mathfrak{v} \in \mathscr{T}^{std}(\mu)$, and $\lambda, \mu \in \Lambda_r^+(n)$.

Notice, in particular, that the Proposition implies that $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is also a cellular basis of $\mathscr{H}_{r,n,}(\mathbf{u})$.

Although we will not pursue this here, we remark that $F_{\mathfrak{t}} = \frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{t}\mathfrak{t}}$ and that these elements give a complete set of pairwise orthogonal primitive idempotents for $\mathscr{H}_{r,n}(\mathbf{u})$. This can be proved by repeating the argument of [Mat04, Theorem 2.15]

Suppose that λ is a multipartition of n and let $S(\lambda)$ be the associated Specht module, or cell module, of $\mathscr{H}_{r,n}(\mathbf{u})$. Thus, $S(\lambda)$ is the free R–module with basis $\{ m_{\mathfrak{s}} \mid \mathfrak{s} \in \mathscr{T}^{std}(\lambda) \}$ and where the action of $\mathscr{H}_{r,n}(\mathbf{u})$ on $S(\lambda)$ is given by

$$a m_{\mathfrak{s}} = \sum_{\mathfrak{u} \in \mathscr{T}^{std}(\lambda)} r_{\mathfrak{s}\mathfrak{u}}(a) m_{\mathfrak{u}},$$

where the scalars $r_{\mathfrak{s}\mathfrak{u}}(a) \in R$ are as in Definition 6.2(c).

It follows directly from Definition 6.2 that $S(\lambda)$ comes equipped with a symmetric bilinear form $\langle\ ,\ \rangle$ which is determined by

$$\langle m_{\mathfrak{s}}, m_{\mathfrak{t}} \rangle m_{\mathfrak{u}\mathfrak{v}} \equiv m_{\mathfrak{u}\mathfrak{s}} m_{\mathfrak{t}\mathfrak{v}} \pmod{\mathscr{H}_{r,n}^{\rhd\lambda}},$$

for $\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v} \in \mathscr{T}^{std}(\lambda)$. Let $G(\lambda) = \det\big(\langle m_{\mathfrak{s}}, m_{\mathfrak{t}} \rangle\big)$, for $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}(\lambda)$, be the Gram determinant of this form. So $\mathcal{G}(\lambda)$ is well–defined up to a unit in R.

Corollary 6.9. *Suppose that R is a field with $\operatorname{char} R > n$ and that $\mathbf{u}$ is generic for $\mathscr{H}_{r,n}(\mathbf{u})$. Let λ be a multipartition of n. Then*

$$\mathcal{G}(\lambda) = \prod_{\mathfrak{t} \in \mathscr{T}^{std}(\lambda)} \gamma_{\mathfrak{t}}.$$

Proof. Fix $\mathfrak{t} \in \mathscr{T}^{std}(\lambda)$. Then Specht module $S(\lambda)$ is isomorphic to the submodule of $\mathscr{H}_{r,n}/\mathscr{H}_{r,n}^{\rhd\lambda}$ which is spanned by $\{ m_{\mathfrak{s}\mathfrak{t}} + \mathscr{H}_{r,n}^{\rhd\lambda} \mid \mathfrak{s} \in \mathscr{T}^{std}(\lambda) \}$, where the isomoprhism is given by $\theta : \mathscr{H}_{r,n}/\mathscr{H}_{r,n}^{\rhd\lambda} \longrightarrow S(\lambda); m_{\mathfrak{s}\mathfrak{t}} + \mathscr{H}_{r,n}^{\rhd\lambda} \mapsto m_{\mathfrak{s}}$. Let $f_{\mathfrak{s}} = \theta(f_{\mathfrak{s}\mathfrak{t}})$. Then $\{ f_{\mathfrak{s}} \mid \mathfrak{s} \in \mathscr{T}^{std}(\lambda) \}$ is a basis of $S(\lambda)$ and the transistion matrix between the two bases $\{m_{\mathfrak{s}}\}$ and $\{f_{\mathfrak{s}}\}$ of $S(\lambda)$ is unitriangular by Lemma 6.6. Consequently, $\mathcal{G}(\lambda) = \det\big(\langle f_{\mathfrak{s}}, f_{\mathfrak{t}} \rangle\big)$, where $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}(\lambda)$. However, it follows from the multiplication formulae in Proposition 6.8 that $\langle f_{\mathfrak{s}}, f_{\mathfrak{t}} \rangle = \delta_{\mathfrak{s}\mathfrak{t}} \gamma_{\mathfrak{t}}$; see the proof of [Mat04, Theorem 2.11] for details. Hence the result. □

Consequently, in order to compute $\mathcal{G}(\lambda)$ it is sufficent to determine γ_t, for all $\mathfrak{t} \in \mathscr{T}^{std}(\lambda)$. It is possible to give an explicit closed formula for $\gamma_{\mathfrak{t}}$ (cf. [Mat04, (2.8)]), however, the following recurrence relation is easier to check and sufficient for our purposes.

Given two standard λ–tableaux $\mathfrak{s}$ and $\mathfrak{t}$ write $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if $\mathfrak{s}_k \trianglerighteq \mathfrak{t}_k$, for $1 \le k \le n$. Let $\mathfrak{t}^\lambda$ be the unique standard λ–tableaux such that $\mathfrak{t}^\lambda \trianglerighteq \mathfrak{s}$ for all $\mathfrak{s} \in \mathscr{T}^{std}(\lambda)$. If $\mathfrak{s} \trianglerighteq \mathfrak{t}$ and $\mathfrak{s} \ne \mathfrak{t}$ then we write $\mathfrak{s} \triangleright \mathfrak{t}$.

Lemma 6.10. *Suppose that R is generic for $\mathscr{H}_{r,n}(\mathbf{u})$ and that $\operatorname{char} R > n$. Let λ be a multipartition of n.*

a) $\gamma_{\mathfrak{t}^\lambda} = \prod_{1\le t\le r} \prod_{i\ge 1} (\lambda_i^{(t)})! \cdot \prod_{1\le s<t\le r} \prod_{\substack{i,j\ge 1\\ 1\le j\le \lambda_i^{(s)}}} (j - i + u_s - u_t).$

b) Suppose that $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}(\lambda)$ such that $\mathfrak{s} \triangleright \mathfrak{t}$ and $\mathfrak{s} = S_k\mathfrak{t}$, for some k. Then $\gamma_{\mathfrak{t}} = \frac{(c_{\mathfrak{s}}(k)-c_{\mathfrak{t}}(k)+1)(c_{\mathfrak{s}}(k)-c_{\mathfrak{t}}(k)-1)}{(c_{\mathfrak{s}}(k)-c_{\mathfrak{t}}(k))^2}\gamma_{\mathfrak{s}}$.

Proof. Part (a) follows easily by induction on n. Part (b) follows using arguments similar to [JM00, Cor. 3.14 and Prop. 3.19] □

We remark that the arguments of [JM00, 3.30-3.37] can now be adapted to give a closed formula for $\mathcal{G}(\lambda)$. The final result is that

$$\mathcal{G}(\lambda) = \prod_{\nu\in\Lambda_r^+(n)} g_{\lambda\nu}^{|\mathscr{T}^{std}(\lambda)|},$$

where $g_{\lambda\nu}$ is a product of terms of the form $\big(c_{\mathfrak{t}^\lambda}(k) - c_{\mathfrak{t}^\nu}(l)\big)^{\pm1}$, where these terms are determined in exactly the same way as in [JM00, Defn 3.36]. As we do not need the precise formula we leave the details to the interested reader.

Theorem 6.11. *Suppose that R is a field and that $\mathbf{u} \in R^r$. Then $\mathscr{H}_{r,n}(\mathbf{u})$ is (split) semisimple if and only if $\operatorname{char} R > n$ and $\mathbf{u}$ is generic for $\mathscr{H}_{r,n}(\mathbf{u})$.*

Proof. First, note that because $\mathscr{H}_{r,n}(\mathbf{u})$ is cellular, it is semisimple if and only if it is split semisimple; see, for example, [Mat99, Cor 2.21].

Next, suppose that $\operatorname{char} R > n$ and that $\mathbf{u}$ is generic for $\mathscr{H}_{r,n}(\mathbf{u})$. Then $\mathcal{G}(\lambda) \ne 0$ for all $\lambda \in \Lambda_r^+(n)$ by Lemma 6.10. Consequently, for each $\lambda \in \Lambda_r^+(n)$ the Specht module $S(\lambda)$ is irreducible. Hence, by [Mat99, Cor 2.21] again, $\mathscr{H}_{r,n}(\mathbf{u})$ is semisimple.

To prove the converse, let $\lambda = ((n),(0),\dots,(0)) \in \Lambda_r^+(n)$ and set $m_\lambda = m_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$; more explicitly,

$$m_\lambda = \sum_{w\in\mathfrak{S}_n} T_w \cdot \prod_{t=2}^{r}\prod_{k=1}^{n} (Y_k - u_t).$$

It is easy to see that $T_\sigma m_\lambda = m_\lambda = m_\lambda T_\sigma$, for any $\sigma \in \mathfrak{S}_n$. It also follows from Lemma 6.6 that $Y_k m_\lambda = c_{\mathfrak{t}^\lambda}(k) m_\lambda = m_\lambda Y_k$. Hence, $\mathscr{H}_{r,n}(\mathbf{u}) m_\lambda \mathscr{H}_{r,n}(\mathbf{u}) = R m_\lambda$ and

$$m_\lambda^2 = n! \prod_{t=2}^{r}\prod_{d=0}^{n-1} (u_1 + d - u_t) \cdot m_\lambda.$$

If $\operatorname{char} R \le n$ then $n! = 0$ in R so that Rm_λ is a nilpotent ideal in $\mathscr{H}_{r,n}(\mathbf{u})$, so $\mathscr{H}_{r,n}(\mathbf{u})$ is not semisimple. On the other hand if $\mathbf{u}$ is not generic for $\mathscr{H}_{r,n}(\mathbf{u})$ then $u_i - u_j = d1_R$ where $d \in \mathbb{Z}$ and $|d| < n$, for some $i \ne j$. By renumbering $u_1,\dots,u_r$, if necessary, we see that Rm_λ is a nilpotent ideal. Hence, if either $\operatorname{char} R \le n$, or if $\mathbf{u}$ is not generic for $\mathscr{H}_{r,n}(\mathbf{u})$, then $\mathscr{H}_{r,n}(\mathbf{u})$ is not semisimple. □

7. A cellular basis of $\mathscr{W}_{r,n}(\mathbf{u})$

In this section, we will construct a cellular basis for $\mathscr{W}_{r,n} = \mathscr{W}_{r,n}(\mathbf{u})$ using the cellular bases of the algebras $\mathscr{H}_{r,n-2f}$, for $0 \le f \le \lfloor \frac{n}{2} \rfloor$, together with a series of filtrations of $\mathscr{W}_{r,n}$. Our construction of a cellular basis of $\mathscr{W}_{r,n}$ is modelled on Enyang's work [Eny04] for the Brauer and BMW algebras.

Throughout this section we assume that R is a commutative ring in which 2 invertible and that Ω is $\mathbf{u}$–admissible.

Before we begin we need to fix some notation. Recall that the set $\{S_1, \dots, S_{n-1}\}$ generates a subalgebra of $\mathscr{W}_{r,n}$ which is isomorphic to the group ring of $\mathfrak{S}_n$. For each permutation $w \in \mathfrak{S}_n$ we defined the corresponding braid diagram $\gamma(w)$ in section 2; we now set $S_w = B_{\gamma(w)}$. Equivalently, if $w = (i_1, i_1+1)\dots(i_k, i_k+1)$, where $1 \le i_j < n$ for all j, then $S_w = S_{i_1}\dots S_{i_k}$. Then $\{\, S_w \mid w \in \mathfrak{S}_n \,\}$ is a basis for the subalgebra of $\mathscr{W}_{r,n}$ generated by $\{S_1, \dots, S_{n-1}\}$.

Next, suppose that f is an integer with $0 \le f \le \lfloor \frac{n}{2} \rfloor$. It follows from Theorem 6.1 that we can identity $\mathscr{H}_{r,n-2f}$ with the subalgebra of $\mathscr{H}_{r,n}$ generated by Y_i and T_j, where $1 \le i \le n-2f$ and $1 \le j \le n-2f-1$. Similarly, by Proposition 5.5, we can identify $\mathscr{W}_{r,n-2f}$ with the subalgebra of $\mathscr{W}_{r,n}$ generated by X_i, S_j and E_j, where $1 \le i \le n-2f$ and $1 \le j \le n-2f-1$.

Definition 7.1. Suppose $0 \le f < \lfloor \frac{n}{2} \rfloor$.

a) Let $\mathcal{E}_f = \mathscr{W}_{r,n-2f} E_1 \mathscr{W}_{r,n-2f}$ be the two-sided ideal of $\mathscr{W}_{r,n-2f}$ generated by E_1.

b) Let $E^f = E_{n-1}E_{n-3}\cdots E_{n-2f+1}$ and let $\mathscr{W}_{r,n}^f = \mathscr{W}_{r,n} E^f \mathscr{W}_{r,n}$ be the two-sided ideal of $\mathscr{W}_{r,n}$ generated by E^f.

If $f = \lfloor \frac{n}{2} \rfloor$ then we set $\mathscr{H}_{r,n-2f} = R$ and $\mathscr{W}_{r,n}^{f+1} = 0$.

Note that this gives a filtration of $\mathscr{W}_{r,n}$ by two–sided ideals:

$$\mathscr{W}_{r,n} = \mathscr{W}_{r,n}^0 \supset \mathscr{W}_{r,n}^1 \supset \dots \supset \mathscr{W}_{r,n}^{\lfloor \frac{n}{2} \rfloor} \supset \mathscr{W}_{r,n}^{\lfloor \frac{n}{2} \rfloor + 1} = 0.$$

For $0 \le f \le \lfloor \frac{n}{2} \rfloor$ let $\pi_f : \mathscr{W}_{r,n}^f \longrightarrow \mathscr{W}_{r,n}^f / \mathscr{W}_{r,n}^{f+1}$ be the corresponding projection map of $\mathscr{W}_{r,n}$–bimodules.

Proposition 7.2. *Suppose that $0 \le f < \lfloor \frac{n}{2} \rfloor$. Then there is a unique R–algebra isomorphism $\varepsilon_f : \mathscr{H}_{r,n-2f} \cong \mathscr{W}_{r,n-2f}/\mathcal{E}_f$ such that*

$$\varepsilon_f(T_i) = S_i + \mathcal{E}_f \text{ and } \varepsilon_f(Y_j) = X_j + \mathcal{E}_f,$$

for $1 \le i < n-2f$ and $1 \le j \le n-2f$.

Proof. We first show that $\mathscr{W}_{r,n-2f}/\mathcal{E}_f$ is a free R–module of rank $r^{n-2f}(n-2f)!$. It follows from the multiplication formulae for Brauer diagrams that an r-regular monomial $X^\alpha B_\gamma X^\beta$ in $\mathscr{W}_{r,n-2f}$ belongs to $\mathcal{E}_f$ whenever γ has a horizontal edge (equivalently, $\gamma \ne \gamma(w)$ for some $w \in \mathfrak{S}_{n-2f}$). If $\gamma = \gamma(w)$, for some $w \in \mathfrak{S}_{n-2f}$, then $B_\gamma = S_w$ and γ contains no horizonal edges, so the definition of regularity (Definition 2.9), forces $\beta = 0$. So, by Theorem 5.4, $\mathscr{W}_{r,n-2f}/\mathcal{E}_f$ is spanned by the elements $\{\, X^\alpha S_w + \mathcal{E}_f \mid 0 \le \alpha_i < r, \text{ for } 1 \le i \le n-2f, \text{ and } w \in \mathfrak{S}_{n-2f} \,\}$. Note that this set contains $r^{n-2f}(n-2f)!$ elements.

To see that the elements at the end of the last paragraph are linearly independent we use the seminormal representations from section 4. Using the arguments and the notation from the proof of Theorem 5.4, it is enough to show that $\dim_{\mathbb{R}} \mathscr{W}_{\mathbb{R}}(\mathbf{u}')/\mathcal{E}_f \ge r^{n-2f}(n-2f)!$. Now a seminormal representation $\Delta(\lambda)$ of $\mathscr{W}_{\mathbb{R}}(\mathbf{u}')$ is a representation of $\mathscr{W}_{\mathbb{R}}(\mathbf{u}')/\mathcal{E}_f$ if and only if $\mathcal{E}_f\Delta(\lambda) = 0$, which happens if and only if λ is a multipartition of $n-2f$. Therefore, by the arguments of section 5, $\dim_{\mathbb{R}} \mathscr{W}_{\mathbb{R}}(\mathbf{u}')/\mathcal{E}_f \ge r^{n-2f}(n-2f)!$. Hence, by the arguments used in the proof of Theorem 5.4 (compare, Theorem 6.1), the elements above are a basis of $\mathscr{W}_{r,n-2f}/\mathcal{E}_f$ and, consequently, $\mathscr{W}_{r,n-2f}/\mathcal{E}_f$ is free as an R–module of rank $r^{n-2f}(n-2f)!$ as claimed.

Inspecting the relations of $\mathscr{H}_{r,n-2f}$ and $\mathscr{W}_{r,n-2f}$ shows that there is a unique homomorphism $\varepsilon_f : \mathscr{H}_{r,n-2f} \longrightarrow \mathscr{W}_{r,n-2f}/\mathcal{E}_f$ such that $\varepsilon_f(T_i) = S_i + \mathcal{E}_f$ and $\varepsilon_f(Y_j) = X_j + \mathcal{E}_f$. To see that ε_f is an isomorphism observe that ε_f maps the generators of $\mathscr{H}_{r,n-2f}$ to the generators of $\mathscr{W}_{r,n-2f}/\mathcal{E}_f$. Hence, it is an isomorphism with inverse determined by $\varepsilon_f^{-1}(S_i + \mathcal{E}_f) = T_i$ and $\varepsilon_f^{-1}(X_j + \mathcal{E}_f) = Y_i$, for $1 \le i < n-2f$ and $1 \le j \le n-2f$. □

For convenience we let $\mathbb{N}_r = \{0, 1, \dots, r-1\}$ and define $\mathbb{N}_r^{(f)}$ to be the set of n–tupples $\kappa = (k_1, \dots, k_n)$ such that $k_i \in \mathbb{N}_r$ and $k_i \neq 0$ only for $i = n-1, n-3, \dots, n-2f+1$. Thus, if $\kappa \in \mathbb{N}_r^{(f)}$ then $X^\kappa = X_{n-1}^{k_{n-1}} X_{n-3}^{k_{n-3}} \dots X_{n-2f+1}^{k_{n-2f+1}} \in \mathscr{W}_{r,n}$.

Lemma 7.3. *Suppose that $0 \le f \le \lfloor \frac{n}{2} \rfloor$ and that $\kappa \in \mathbb{N}_r^{(f)}$. Then $E^f X^\kappa \mathcal{E}_f \subset \mathscr{W}_{r,n}^{f+1}$.*

Proof. As $E^{f+1} = E^f E_{n-2f-1}$, this follows because $\mathcal{E}_f = \mathscr{W}_{r,n-2f} E_{n-2f-1} \mathscr{W}_{r,n-2f}$ and every element of $\mathscr{W}_{r,n-2f}$ commutes with $E^f X^\kappa$. □

Combining the last two results we have a well-defined R-module homomorphism $\sigma_f : \mathscr{H}_{r,n-2f} \longrightarrow \mathscr{W}_{r,n}^f / \mathscr{W}_{r,n}^{f+1}$, for each integer f, with $0 \le f \le \lfloor \frac{n}{2} \rfloor$, given by

$$\sigma_f(h) = E^f \varepsilon_f(h) + \mathscr{W}_{r,n}^{f+1},$$

for $h \in \mathscr{H}_{r,n-2f}$.

We will need the following subgroups in order to understand the ideals $\mathscr{W}_{r,n}^f$.

Definition 7.4. Suppose that $0 \le f \le \lfloor \frac{n}{2} \rfloor$. Let B_f be the subgroup of $\mathfrak{S}_n$ generated by $\{S_{n-1}, S_{n-2}S_{n-1}S_{n-3}S_{n-2}, \cdots, S_{n-2f+2}S_{n-2f+1}S_{n-2f+3}S_{n-2f+2}\}$.

The group B_f is isomorphic to the hyeroctahedral group $\mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_f$, a Coxeter group of type B_f.

Given an integer f, with $0 \le f \le \lfloor \frac{n}{2} \rfloor$, let $\tau = \big((n-2f), (2^f)\big)$ and define

$$D_f = \left\{ d \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} \mathfrak{t}^\tau d = (\mathfrak{t}_1, \mathfrak{t}_2) \text{ is a row standard } \tau\text{-tableau and the} \\ \text{first column of } \mathfrak{t}_2 \text{ is increasing from top to bottom} \end{array} \right\}.$$

The following result is equivalent to [Eny04, Prop. 3.1]. (Enyang considers a subgroup of $\mathfrak{S}_n$ which is conjugate to B_f.)

Lemma 7.5. *Suppose that $0 \le f \le \lfloor \frac{n}{2} \rfloor$. Then D_f is a complete set of right coset representatives for $\mathfrak{S}_{n-2f} \times B_f$ in $\mathfrak{S}_n$.*

The point of introducing the subgroup B_f is the following.

Lemma 7.6. *Suppose that $0 \le f \le \lfloor \frac{n}{2} \rfloor$, $w \in \mathfrak{S}_{n-2f}$ and that $b \in B_f$. Then $S_w E^f = E^f S_w$ and $E^f S_b = E^f = S_b E^f$.*

Proof. Both claims follow easily using the relations of $\mathscr{W}_{r,n}$. When proving that $E^f S_b = E^f$ the untwisting relations $E_j E_{j\pm1} E_j = E_j$ are useful. □

Motivated by the definition of the elements $m_{\mathfrak{st}} \in \mathscr{H}_{r,n-2f}$ from the previous section, and by the work of Enyang [Eny04], we make the following definition.

Definition 7.7. Suppose that $0 \le f \le \lfloor \frac{n}{2} \rfloor$ and $\lambda \in \Lambda_r^+(n-2f)$. Then for each pair $(\mathfrak{s}, \mathfrak{t})$ of standard λ–tableaux define

$$M_{\mathfrak{st}} = S_{d(\mathfrak{s})^{-1}} \cdot \prod_{s=2}^{r} \prod_{i=1}^{|\lambda^{(1)}|+\dots+|\lambda^{(s-1)}|} (X_i - u_s) \sum_{w \in \mathfrak{S}_\lambda} S_w \cdot S_{d(\mathfrak{t})}.$$

We remark that we will not ever really use this explcit formula for the elements $M_{\mathfrak{st}}$. In what follows all that we need is a family of elements $\{M_{\mathfrak{st}}\}$ in $\mathscr{W}_{r,n}$ which are related to some cellular basis of $\mathscr{H}_{r,n-2f}$ as in Lemma 7.8(d) below.

The following result follows easily using therelations of $\mathscr{W}_{r,n}$, Lemma 7.6 and the definitions.

Lemma 7.8. *Suppose that $0 \le f \le \lfloor \frac{n}{2} \rfloor$, $\lambda \in \Lambda_r^+(n-2f)$ and that $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}(\lambda)$. Then:*

a) $E^f M_{\mathfrak{st}} = M_{\mathfrak{st}} E^f \in \mathscr{W}_{r,n}^f$.
b) *If* $\kappa \in \mathbb{N}_r^{(f)}$ *then* $M_{\mathfrak{st}} X^\kappa = X^\kappa M_{\mathfrak{st}}$.
c) *If* w *is a permutation of* $\{n-2f+1, \dots, n\}$ *then* $M_{\mathfrak{st}} S_w = S_w M_{\mathfrak{st}}$. *In particular,* $M_{\mathfrak{st}} S_w = S_w M_{\mathfrak{st}}$ *if* $w \in B_f$.
d) *We have* $\sigma_f(m_{\mathfrak{st}}) = \pi_f(E^f M_{\mathfrak{st}})$.

The filtration of $\mathscr{W}_{r,n}$ given by the ideals $\mathscr{W}_{r,n}^f$ is still too coarse to be cellular. We now define $\mathscr{W}_{r,n}^{\trianglerighteq\lambda}$ to be the two–sided ideal of $\mathscr{W}_{r,n}$ generated by $\mathscr{W}_{r,n}^{f+1}$ and the elements $\{\, E^f M_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}(\mu) \text{ and } \mu \in \Lambda_r^+(n-2f) \text{ with } \mu \trianglerighteq \lambda \,\}$, and we set $\mathscr{W}_{r,n}^{\triangleright\lambda} = \sum_{\mu \triangleright \lambda} \mathscr{W}_{r,n}^{\trianglerighteq\mu}$, where in the sum $\mu \in \Lambda_r^+(n-2f)$. Observe that $\mathscr{W}_{r,n}^{f+1} \subseteq \mathscr{W}_{r,n}^{\triangleright\lambda} \subseteq \mathscr{W}_{r,n}^f$ and that $\mathscr{W}_{r,n}^{\trianglerighteq\lambda} \subset \mathscr{W}_{r,n}^{\triangleright\mu}$ whenever $\lambda \triangleright \mu$. Consequently, the ideals $\{\mathscr{W}_{r,n}^{\triangleright\lambda}\}$ give a refinement of the filtration of $\mathscr{W}_{r,n}$ by the ideals $\{\mathscr{W}_{r,n}^f\}$.

Definition 7.9. Suppose that $\mathfrak{s} \in \mathscr{T}^{std}(\lambda)$. We define $\Delta_{\mathfrak{s}}(f,\lambda)$ to R-submodule of $\mathscr{W}_{r,n}^{\trianglerighteq\lambda} / \mathscr{W}_{r,n}^{\triangleright\lambda}$ which is spanned by the elements

$$\{\, E^f M_{\mathfrak{st}} X^\kappa S_d + \mathscr{W}_{r,n}^{\triangleright\lambda} \mid (\mathfrak{t}, \kappa, d) \in \delta(f,\lambda) \,\},$$

where $\delta(f,\lambda) = \{\, (\mathfrak{t}, \kappa, d) \mid \mathfrak{t} \in \mathscr{T}^{std}(\lambda), \kappa \in \mathbb{N}_r^{(f)} \text{ and } d \in D_f \,\}$.

We will see below that the spanning set in the definition is a basis of $\Delta_{\mathfrak{s}}(f,\lambda)$.

Before we begin studying the modules $\Delta_{\mathfrak{s}}(f,\lambda)$ it is convenient to define a degree function on $\mathscr{W}_{r,n}$. Recall from Theorem 5.4 that the set of r–regular monomials is a basis of $\mathscr{W}_{r,n}$.

Definition 7.10. Suppose that $a = \sum r_{\alpha\gamma\beta} X^\alpha B_\gamma X^\beta \in \mathscr{W}_{r,n}$, where each of the monomials in the sum is r–regular. Then the degree of a is the integer

$$\deg a = \max \Big\{ \sum_{i=1}^n (\alpha_i + \beta_i) \,\Big|\, r_{\alpha\gamma\beta} \neq 0 \text{ for some } \gamma \in \mathcal{B}(n) \Big\}.$$

In particular, $\deg S_i = \deg E_i = 0$, for $1 \le i < n$, and $\deg X_j = 1$, for $1 \le j \le n$. We note that the defining relations of $\mathscr{W}_{r,n}$ imply that $\deg(ab) \le \deg(a) + \deg(b)$, for all $a, b \in \mathscr{W}_{r,n}$.

Lemma 7.11. *Suppose that $1 \le k < n$ and that $k \ge 0$. Then $E_j X_j^k E_j = E_j \omega_j^{(k)}$, where $\omega_j^{(k)}$ is a central element in $\mathscr{W}_{r,j-1}$ with $\deg \omega_j^{(k)} < k$.*

Proof. By Lemma 4.15, $E_j X_j^k E_j = E_j \omega_j^{(k)}$, where $\omega_j^{(k)}$ is a central element in $\mathscr{W}_{r,j-1}$. We prove that $\deg \omega_j^{(k)} < k$ by induction on j.

If $j = 1$ there is nothing to prove because $\omega_1^{(k)} \in R$ by relation (2.1)(f). If $j > 1$ then

$$\begin{aligned} E_j X_j^k E_j &= (-1)^k E_j X_{j+1}^k E_j = (-1)^k E_j S_{j-1} X_{j+1}^k S_{j-1} E_j \\ &= (-1)^k E_j E_{j-1} S_j X_{j+1}^k S_j E_{j-1} E_j \end{aligned}$$

By Lemma 2.3, $S_jX_{j+1}^kS_j = X_j^k + X$, where $\deg X < k$. Now, $\deg(E_kE_{k-1}XE_{k-1}E_k) \le \deg X$, since $\deg(ab) \le \deg(a) + \deg(b)$. We also have

$$\begin{aligned} E_jE_{j-1}X_j^kE_{j-1}E_j &= (-1)^kE_jE_{j-1}X_{j-1}^kE_{j-1}E_j = (-1)^k\omega_{j-1}^{(k)}E_jE_{j-1}E_j \\ &= (-1)^k\omega_{j-1}^{(k)}E_j. \end{aligned}$$

By induction $\deg \omega_{j-1}^{(k)} < k$, so this completes the proof of the Lemma. □

Given integers j and k, with $1 \le j, k \le n$, let $E_{j,k} = B_\gamma$ where γ is the Brauer diagram with horizontal edges $\{j, k\}$ and $\{\overline{j}, \overline{k}\}$, and with all other edges being vertical. Thus, if $j < k$ then $E_{j,k} = S_{(j,k)}E_{k-1}S_{(k,j)}$; more generally, $S_wE_iS_{w^{-1}} = E_{(i)w^{-1},(i+1)w^{-1}}$, for all $w \in \mathfrak{S}_n$. Finally, note that $E_i = E_{i,i+1}$.

Until further notice we fix an integer f, with $0 \le f \le \lfloor \frac{n}{2} \rfloor$, a multipartition $\lambda \in \Lambda_r^+(n-2f)$ and a standard λ–tableau $\mathfrak{s}$ and consider $\Delta(f,\lambda) = \Delta_{\mathfrak{s}}(f,\lambda)$. The next two Lemmas show that $\Delta(f,\lambda)$ is a $\mathscr{W}_{r,n}$–submodule of $\mathscr{W}_{r,n}^{\trianglerighteq\lambda}/\mathscr{W}_{r,n}^{\triangleright\lambda}$ and that the action of $\mathscr{W}_{r,n}$ on $\Delta(f,\lambda)$ does not depend on $\mathfrak{s}$.

If $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{N}_r^{(f)}$ we set $|\kappa| = \kappa_{n-1} + \kappa_{n-3} + \dots + \kappa_{n-2f+1} = \deg X^\kappa$.

Lemma 7.12. *Suppose that $\mathfrak{t} \in \mathscr{T}^{std}(\lambda)$ and $d \in D_f$. Then for each $h \in \{ S_I, E_i, X_j \mid 1 \le i < n \text{ and } 1 \le j \le n \}$ there exist scalars $r'_{\mathfrak{v}\rho d}(h) \in R$, which do not depend on $\mathfrak{s}$, such that*

$$E^fM_{\mathfrak{s}\mathfrak{t}}S_d \cdot h \equiv \sum_{\substack{(\mathfrak{v},\rho,e)\in\delta(f,\lambda)\\ |\rho|\le\deg h}} r'_{\mathfrak{v}\rho e}(h)E^fM_{\mathfrak{s}\mathfrak{v}}X^\rho S_e \pmod{\mathscr{W}_{r,n}^{\triangleright\lambda}}.$$

Proof. We consider the three cases $h = S_i$, $h = E_i$ and $h = X_j$ in turn.

Case 1. $h = S_i$, where $1 \le i < n$:
Now, $S_dS_i = S_{d(i,i+1)}$ and by Lemma 7.5 we can write $d(i,i+1) = abe$ where $a \in \mathfrak{S}_{n-2f}$, $b \in B_f$ and $e \in D_f$; so $S_dS_i = S_aS_bS_e$. By parts (b) and (c) of Lemma 7.8, respectively, we have

$$E^fM_{\mathfrak{s}\mathfrak{t}}S_a = E^fM_{\mathfrak{s}\mathfrak{t}}S_a \equiv E^f\varepsilon_f(m_{\mathfrak{s}\mathfrak{t}})S_a \equiv E^f\varepsilon_f(m_{\mathfrak{s}\mathfrak{t}}T_a) \pmod{\mathscr{W}_{r,n}^{\triangleright\lambda}},$$

since $\mathscr{W}_{r,n}^{f+1} \subseteq \mathscr{W}_{r,n}^{\triangleright\lambda}$. As $m_{\mathfrak{s}\mathfrak{t}}$ is a cellular basis element for $\mathscr{H}_{r,n-2f}$, we can write $m_{\mathfrak{s}\mathfrak{t}}T_a$ as a linear combination of terms $m_{\mathfrak{s}\mathfrak{v}}$ plus an element of $\mathscr{H}_{r,n}^{\triangleright\lambda}$. Consequently, $(E^fM_{\mathfrak{s}\mathfrak{t}} + \mathscr{W}_{r,n}^{\triangleright\lambda})S_a$ can be written in the desired form. Hence, we may now assume that $a = 1$.

To complete this case, observe that if $\mathfrak{v} \in \mathscr{T}^{std}(\lambda)$ then, by Lemma 7.8(c) and Lemma 7.6, $E^fM_{\mathfrak{s}\mathfrak{v}}S_bS_e = E_fS_bM_{\mathfrak{s}\mathfrak{v}}S_e = E^fM_{\mathfrak{s}\mathfrak{v}}S_e$ as required.

Case 2. $h = E_i$, where $1 \le i < n$:
We have to consider the product $E^fM_{\mathfrak{s}\mathfrak{t}}S_dE_i$. Let $j = (i)d^{-1}$ and $k = (i+1)d^{-1}$. Then $S_dE_i = E_{j,k}S_d$ so that $E^fM_{\mathfrak{s}\mathfrak{t}}S_dE_i = E^fM_{\mathfrak{s}\mathfrak{t}}E_{j,k}S_d$. By Case 1 we may assume that $d = 1$. We can also assume that $j < k$ since $E_{j,k} = E_{k,j}$. So we need to show that $E^fM_{\mathfrak{s}\mathfrak{t}}E_{j,k} + \mathscr{W}_{r,n}^{\triangleright\lambda}$ has the required form. There are three cases to consider.

(1) First, suppose that $j < k \le n-2f$. Then $E_{j,k} \in \mathscr{W}_{r,n-2f}$, so that $E_{j,k} \in \mathcal{E}_f$. Hence, $E^fM_{\mathfrak{s}\mathfrak{t}}S_dE_i \in \mathscr{W}_{r,n}^{f+1} \subseteq \mathscr{W}_{r,n}^{\triangleright\lambda}$ by Lemma 7.3, so the Lemma is true when $j < k \le n-2f$.

(2) Next, suppose that $j \le n-2f < k$. An easy exercise in multiplying Brauer diagrams shows that

$$E^fE_{j,k} = \begin{cases} E^fS_{(j,k+1)}, & \text{if } n-k \text{ is even,} \\ E^fS_{(j,k-1)}, & \text{if } n-k \text{ is odd.} \end{cases}$$

As $E^fM_{\mathfrak{s}\mathfrak{t}}E_{j,k} = M_{\mathfrak{s}\mathfrak{t}}E^fE_{j,k}$ we again deduce the result from Case 1.

(3) Finally, suppose that $n-2f<j<k$. Then $M_{\mathfrak{s}\mathfrak{t}}E_{j,k}=E_{j,k}M_{\mathfrak{s}\mathfrak{t}}$ and a Brauer diagram calculation shows that $E^fE_{j,k}=E^fS_w$, where w is a permutation of $\{n-2f+1,\dots,n\}$. Consequently,

$$E^fM_{\mathfrak{s}\mathfrak{t}}E_{j,k}=E^fE_{j,k}M_{\mathfrak{s}\mathfrak{t}}=E^fS_wM_{\mathfrak{s}\mathfrak{t}}=E^fM_{\mathfrak{s}\mathfrak{t}}S_w,$$

where the last equality follows from Lemma 7.8(c). Once again, we are done by Case 1.

Case 3. $h=X_j$**, where** $1\le j\le n$:
If $r=1$ then X_i can be written as a linear combination of words in the generators E_i and S_i, so we may assume that $r>1$. It follows from the skein relations that $S_dX_j=X_{(j)d}+B$, for some $B\in\mathscr{B}_n(\omega_0)$. Hence, by Cases 1 and 2 it suffices to show that $E^fM_{\mathfrak{s}\mathfrak{t}}X_i$ can be written in the required form, for $1\le i\le n$. If $i\le n-2f$ then

$$E^fM_{\mathfrak{s}\mathfrak{t}}X_i+\mathscr{W}^{\rhd\lambda}_{r,n}=E^f\sigma(m_{\mathfrak{s}\mathfrak{t}})X_i+\mathscr{W}^{\rhd\lambda}_{r,n}=E^f\sigma(m_{\mathfrak{s}\mathfrak{t}}Y_i)+\mathscr{W}^{\rhd\lambda}_{r,n},$$

so the result follows because $m_{\mathfrak{s}\mathfrak{t}}$ is a cellular basis element of $\mathscr{H}_{r,n-2f}$. If $i>n-2f$ then the result is immediate if $n-i$ is odd. If $n-i$ is even the result follows because $E_iX_{i+1}=-E_iX_i$ by (2.1)(h).

This completes the proof of the Lemma. □

Proposition 7.13. *Suppose that* $(\mathfrak{t},\kappa,d)\in\delta(f,\lambda)$ *and that* $h\in\{\,S_I,E_i,X_j\mid 1\le i<n$ *and* $1\le j\le n\,\}$. *Then there exist scalars* $r_{\mathfrak{v}\rho d}(h)\in R$*, which do not depend on* $\mathfrak{s}$*, such that*

$$E^fM_{\mathfrak{s}\mathfrak{t}}S_d\cdot h\equiv\sum_{\substack{(\mathfrak{v},\rho,e)\in\delta(f,\lambda)\\ |\rho|\le|\kappa|+\deg h}} r_{\mathfrak{v}\rho e}(h)E^fM_{\mathfrak{s}\mathfrak{v}}X^\rho S_e\pmod{\mathscr{W}^{\rhd\lambda}_{r,n}}.$$

Proof. The case $|\kappa|=0$ is precisely Lemma 7.12. We now assume that $|\kappa|>0$ and argue by induction on $|\kappa|$. Once again, we break the proof into three cases.

Case 1. $h=S_i$**, where** $1\le i<n$:
Write $S_dS_i=S_aS_bS_e$, where $a\in\mathfrak{S}_{n-2f}$, $b\in B_f$ and $e\in D_f$. As $E^fM_{\mathfrak{s}\mathfrak{t}}X^\kappa=E^fX^\kappa M_{\mathfrak{s}\mathfrak{t}}$ we may assume that $a=1$ by repeating the argument from the proof Case 1 of Lemma 7.12. By the right handed version of Lemma 2.3, $X^\kappa S_b=S_bX^{\kappa b^{-1}}+X$, where X is a linear combination of monomials of the form $a_1\dots a_k$ with $a_j\in\{S_j,E_j,X_k\}$ and $k<|\kappa|$. For each summand $a_1\dots a_k$ of X we have $k<|\kappa|$ so by induction we can write $(E^fM_{\mathfrak{s}\mathfrak{t}}+\mathscr{W}^{\rhd\lambda}_{r,n})a_1\dots a_l$ in the required form, for $l=1,\dots,k$; consequently, by induction, we can write $(E^fM_{\mathfrak{s}\mathfrak{t}}+\mathscr{W}^{\rhd\lambda}_{r,n})a_1\dots a_kS_e$ in the required form. Hence, we are reduced to showing that $E^fM_{\mathfrak{s}\mathfrak{t}}S_bX^{\kappa b^{-1}}S_e+\mathscr{W}^{\rhd\lambda}_{r,n}$ can be written in the required form. Now, $E^fM_{\mathfrak{s}\mathfrak{t}}S_b=E^fS_bM_{\mathfrak{s}\mathfrak{t}}=E^fM_{\mathfrak{s}\mathfrak{t}}$ by Lemma 7.8(c) and Lemma 7.6. Therefore, using Lemma 7.6 once again,

$$E^fM_{\mathfrak{s}\mathfrak{t}}S_bX^{\kappa b^{-1}}S_e=E^fM_{\mathfrak{s}\mathfrak{t}}X^{\kappa b^{-1}}S_e=M_{\mathfrak{s}\mathfrak{t}}E^fX^{\kappa b^{-1}}S_e=\pm M_{\mathfrak{s}\mathfrak{t}}E^fX^{\kappa'}S_e,$$

where $\kappa'\in\mathbb{N}_r^{(f)}$ because $b\in B^f$ and $E_jX_{j+1}=-E_jX_j$ by the skein relations. Hence, $E^fM_{\mathfrak{s}\mathfrak{t}}S_bX^{\kappa b^{-1}}S_e=\pm E^fM_{\mathfrak{s}\mathfrak{t}}X^{\kappa'}S_e$ and the Proposition is proved when $h=S_i$.

Case 2. $h=E_i$**, where** $1\le i<n$:
As in the proof of Case 2 of Lemma 7.12, we have $E^fM_{\mathfrak{s}\mathfrak{t}}X^\kappa S_dE_i=E^fM_{\mathfrak{s}\mathfrak{t}}X^\kappa E_{j,k}S_d$, where $j=(i)d^{-1}$ and $(i+1)d^{-1}$. Further, as $E_{j,k}=E_{k,j}$ we may assume that $j<k$ and, by Case 1, we may assume that $d=1$. So we need to show that $E^fM_{\mathfrak{s}\mathfrak{t}}X^\kappa E_{j,k}+\mathscr{W}^{\rhd\lambda}_{r,n}$ has the required form. There are three subcases to consider.

(1) If $j < k \le n-2f$ then we may repeat the argument from the proof of Case 2 of Lemma 7.12 to see that $E_{j,k} \in \mathcal{E}_f$, so that $E^f M_{\mathfrak{s}\mathfrak{t}} X^\kappa S_d E_i \in \mathscr{W}_{r,n}^{\rhd\lambda}$.

(2) If $j \le n-2f < k$ then $E_{j,k} = S_u E_{n-2f,n-2f+1} S_v$, where $u \in \mathfrak{S}_{n-2f}$ and v is a permutation of $\mathfrak{S}_{n-2f+1,\dots,n}$. Now,

$$\begin{aligned} E^f M_{\mathfrak{s}\mathfrak{t}} X^\kappa E_{j,k} &= E^f X^\kappa M_{\mathfrak{s}\mathfrak{t}} S_u E_{n-2f,n-2f+1} S_v \\ &\equiv E^f X^\kappa \varepsilon_f(m_{\mathfrak{s}\mathfrak{t}} T_u) E_{n-2f,n-2f+1} S_v \pmod{\mathscr{W}_{r,n}^{\rhd\lambda}}. \end{aligned}$$

Hence, since $m_{\mathfrak{s}\mathfrak{t}}$ is a cellular basis element of $\mathscr{H}_{r,n-2f}$ we may assume that $u = 1$ and, by Case 1, we may assume that $v = 1$. This allows us to reduce to the case $E_{j,k} = E_{n-2f,n-2f+1} = E_{n-2f}$. Now, using relations (d), (f) and (h) from (2.1), we have

$$\begin{aligned} E_{n-2f+1} X_{n-2f+1}^{\kappa_{n-2f+1}} E_{n-2f} &= (-1)^{\kappa_{n-2f+1}} E_{n-2f+1} X_{n-2f}^{\kappa_{n-2f+1}} E_{n-2f} \\ &= (-1)^{\kappa_{n-2f+1}} X_{n-2f}^{\kappa_{n-2f+1}} E_{n-2f+1} E_{n-2f} \\ &= (-1)^{\kappa_{n-2f+1}} X_{n-2f}^{\kappa_{n-2f+1}} E_{n-2f+1} S_{n-2f} S_{n-2f+1} \\ &= (-1)^{\kappa_{n-2f+1}} E_{n-2f+1} X_{n-2f}^{\kappa_{n-2f+1}} S_{n-2f} S_{n-2f+1}. \end{aligned}$$

As $E^f M_{\mathfrak{s}\mathfrak{t}} = M_{\mathfrak{s}\mathfrak{t}} E^f$ and E_{n-2f+1} is a factor of E^f, when $j \le n-2f < k$ the result now follows by Case 1.

(3) Now consider the case when $n-2f < j < k$. We consider only the case when $j \equiv n-1 \pmod 2$ as the argument when $j \equiv n \pmod 2$ is similar. Observe that E_j is a factor of E^f if $j \equiv n-1 \pmod 2$.

Suppose first that $k = j+1$. Then $E_j X_j^{\kappa_j} E_j = E_j \omega_j^{(\kappa_j)}$ by Lemma 7.11, where $\omega_j^{(\kappa_j)}$ is a central element of $\mathscr{W}_{r,j-1}$ with $\deg \omega_j^{(\kappa_j)} < \kappa_j$. Write $E^f = \dot{E}^f E_j$ and $X^\kappa = \dot{X}^\kappa X_j^{\kappa_j}$. Then

$$E^f M_{\mathfrak{s}\mathfrak{t}} X^\kappa E_j = M_{\mathfrak{s}\mathfrak{t}} \dot{E}^f \dot{X}^\kappa E_j X_j^{\kappa_j} E_j = M_{\mathfrak{s}\mathfrak{t}} \dot{E}^f \dot{X}^\kappa E_j \omega_j^{(\kappa_j)} = E^f M_{\mathfrak{s}\mathfrak{t}} \dot{X}^\kappa E_j \omega_j^{(\kappa_j)}.$$

As $\deg \dot{X}^\kappa = |\kappa| - \kappa_j$ and $\deg \omega_j^{(\kappa_j)} < \kappa_j$, the result now follows by writing $\omega_j^{(\kappa_j)}$ as a linear combination of terms of the form $a_1 \dots a_l$ and applying induction to each of the products $E^f M_{\mathfrak{s}\mathfrak{t}} \dot{X}^\kappa E_j a_i \dots a_m$, for $1 \le m \le l$ (compare the proof of Case 1).

Now suppose that $k > j+1$. As E_j divides E^f and $X_j^{\kappa_j}$ divides X^κ we can write $E^f = \dot{E}^f E_j$ and $X^\kappa = \dot{X}^\kappa X_j^{\kappa_j}$. Then $E^f M_{\mathfrak{s}\mathfrak{t}} X^\kappa E_{j,k} = M_{\mathfrak{s}\mathfrak{t}} \dot{E}^f \dot{X}^\kappa E_j X^{\kappa_j} E_{j,k}$. To simplify the notation, if $a \le b$ are integers then set $S_{a,b} = S_a \dots S_b$ and $S_{b,a} = S_{a,b}^{-1}$. Then $E_{j,k} = S_{k-1,j+1} E_j S_{j+1,k-1}$, so

$$\begin{aligned} E_j X^{\kappa_j} E_{j,k} &= E_j X_j^{\kappa_j} S_{k-1,j+1} E_j S_{j+1,k-1} \\ &= S_{k-1,j+2} E_j X_j^{\kappa_j} S_{j+1} E_j S_{j+1,k-1} \\ &= S_{k-1,j+2} E_j S_j X_j^{\kappa_j} S_{j+1} E_j S_{j+1,k-1} \\ &= S_{k-1,j+2} E_j \big(X_{j+1}^{\kappa_j} S_j + X \big) S_{j+1} E_j S_{j+1,k-1}, \end{aligned}$$

where the last equality uses Lemma 2.3, and $X \in \mathscr{W}_{r,j}$ with $\deg X < \kappa_j$. Now, if $\kappa_j = 0$ then $X = 0$. On the other hand, if $\kappa_j > 0$ then

$$M_{\mathfrak{s}\mathfrak{t}} \dot{E}^f \dot{X}^\kappa S_{k-1,j+2} E_j X S_{j+1} E_j S_{j+1,k-1} = E^f M_{\mathfrak{s}\mathfrak{t}} \dot{X}^\kappa S_{k-1,j+2} X S_{j+1} E_j S_{j+1,k-1}.$$

As $\deg \dot{X}^\kappa = |\kappa| - \kappa_j$ we can write this term in the required form using Case 1 and induction. Returning to the first term above, using the relations (2.1) we find that

$$\begin{aligned} S_{k-1,j+2}E_jX_{j+1}^{\kappa_j}S_jS_{j+1}E_jS_{j+1,k-1} &= S_{k-1,j+2}E_jX_{j+1}^{\kappa_j}E_{j+1}E_jS_{j+1,k-1} \\ &= (-1)^{\kappa_j}S_{k-1,j+2}E_jX_{j+2}^{\kappa_j}E_{j+1}E_jS_{j+1,k-1} \\ &= (-1)^{\kappa_j}S_{k-1,j+2}E_jE_{j+1}E_jX_{j+2}^{\kappa_j}S_{j+1,k-1} \\ &= (-1)^{\kappa_j}S_{k-1,j+2}E_jX_{j+2}^{\kappa_j}S_{j+1,k-1}, \end{aligned}$$

since $k \geq j+2$. The contribution of the first term to $E^fM_{\mathfrak{st}}X^\kappa E_{j,k}$ is $(-1)^{\kappa_j}M_{\mathfrak{st}}E^f\dot{X}^\kappa S_{k-1,j+2}X_{j+2}^{\kappa_j}S_{j+1,k-1}$. If $\kappa_j = 0$ then we are done by Case 1. Now suppose that $\kappa_j > 0$. By Case 1 again, $M_{\mathfrak{st}}E^f\dot{X}^\kappa S_{k-1,j+1}$ can be written as a linear combination of terms of the form $E^fM_{\mathfrak{sv}}X^\rho S_e$, with $(\mathfrak{v},\rho,e) \in \delta(f,\lambda)$ and $|\rho| \leq |\kappa| - \kappa_j$. Hence, by induction, $M_{\mathfrak{st}}E^f\dot{X}^\kappa S_{k-1,j+1}X_{j+2}^{\kappa_j} = \big(\dots(M_{\mathfrak{st}}E^f\dot{X}^\kappa S_{k-1,j+1}X_{j+2})\dots\big)X_{j+2}$ can be written as a linear combination of elements of the same form but with $|\rho| \leq |\kappa|$—note that we can apply induction here because, for each of the κ_j multiplications, all of the terms $E^fM_{\mathfrak{sv}}X^\rho S_e$ that we multiply by X_{j+2} have $|\rho| < |\kappa|$. A final appeal to Case 1 shows that we can write $E^fM_{\mathfrak{st}}X^\kappa E_{j,k}$ in the required form.

This completes the proof of the Proposition when $h = E_i$.

Case 3. $h = X_j$**, where** $1 \leq j \leq n$:
As in the proof of Lemma 7.12, we may assume that $r > 1$ and, by the skein relations, $S_dX_j = X_{(j)d} + B$, for some $B \in \mathscr{B}_n(\omega_0)$. Hence, by Cases 1 and 2 it suffices to show that $E^fM_{\mathfrak{st}}X^\kappa\cdot X_i$ can be written in the required form. If $i \leq n-2f$ then

$$E^fM_{\mathfrak{st}}X^\kappa X_i + \mathscr{W}_{r,n}^{\rhd\lambda} = E^fX^\kappa\sigma(m_{\mathfrak{st}})X_i + \mathscr{W}_{r,n}^{\rhd\lambda} = E^fX^\kappa\sigma(m_{\mathfrak{st}}Y_i) + \mathscr{W}_{r,n}^{\rhd\lambda},$$

so the result follows because $\{m_{\mathfrak{st}}\}$ is a cellular basis of $\mathscr{H}_{r,n-2f}$. If $i > n-2f$ and $\kappa_i < r-1$ then $E^fM_{\mathfrak{st}}X^\kappa X_i$ is of the desired form. If $\kappa_i = r-1$ then $X_i^{\kappa_i}X_i = X_i^r$ can be written as a linear combination of r–regular monomials by Theorem 5.4. Hence, using Cases 1 and 2 and induction $E^fM_{\mathfrak{st}}X^\kappa X_i + \mathscr{W}_{r,n}^{\rhd\lambda}$ can be written in the required form.

This completes the proof of the Proposition. □

Recall from (2.2) that $\mathscr{W}^{\mathrm{aff}}$ has a unique anti–automorphism $*\colon \mathscr{W}^{\mathrm{aff}} \longrightarrow \mathscr{W}^{\mathrm{aff}}$ which fixes all of the generators of $\mathscr{W}^{\mathrm{aff}}$. This involution induces an anti–isomorphism of $\mathscr{W}_{r,n}$, which we also call $*$. Thus, $S_i^* = S_i$, $E_i^* = E_i$, $X_j^* = X_j$ and $(ab)^* = b^*a^*$, for $1 \leq i < n$, $1 \leq j \leq n$ and all $a, b \in \mathscr{W}_{r,n}$. Observe that $S_w^* = S_{w^{-1}}$, for $w \in \mathfrak{S}_n$, and that $M_{\mathfrak{st}}^* = M_{\mathfrak{ts}}$.

Proposition 7.14. *Suppose* $0 \leq f \leq \lfloor\frac{n}{2}\rfloor$ *and* $\lambda \in \Lambda_r^+(n-2f)$. *Then* $\mathscr{W}_{r,n}^{\trianglerighteq\lambda}/\mathscr{W}_{r,n}^{\rhd\lambda}$ *is spanned by the elements*

$$\{\, S_e^*X^\rho E^fM_{\mathfrak{st}}X^\kappa S_d + \mathscr{W}_{r,n}^{\rhd\lambda} \mid (\mathfrak{t},\kappa,d), (\mathfrak{s},\rho,e) \in \delta(f,\lambda)\,\}.$$

Proof. Let W be the R–submodule of $\mathscr{W}_{r,n}^{\trianglerighteq\lambda}/\mathscr{W}_{r,n}^{\rhd\lambda}$ spanned by the elements in the statement of the Proposition. As the generators $\{E^fM_{\mathfrak{st}} + \mathscr{W}_{r,n}^{\rhd\lambda}\}$ of $\mathscr{W}_{r,n}^{\trianglerighteq\lambda}/\mathscr{W}_{r,n}^{\rhd\lambda}$ are contained in W, and $W \subseteq \mathscr{W}_{r,n}^{\trianglerighteq\lambda}/\mathscr{W}_{r,n}^{\rhd\lambda}$, it suffices to show that W is a $\mathscr{W}_{r,n}$–bimodule.

First, by Proposition 7.13, W is closed under right multiplication by elements of $\mathscr{W}_{r,n}$. To see that W is also closed under left multiplication by elements of $\mathscr{W}_{r,n}$ note that $(\mathscr{W}_{r,n}^{\rhd\lambda})^* = \mathscr{W}_{r,n}^{\rhd\lambda}$ as the set of generators for $\mathscr{W}_{r,n}^{\rhd\lambda}$ is invariant under $*$. Also, $(E^fM_{\mathfrak{st}})^* = M_{\mathfrak{ts}}(E^f)^* = M_{\mathfrak{ts}}E^f = E^fM_{\mathfrak{ts}}$. Therefore, if $a \in \mathscr{W}_{r,n}$ then

$$a(S_e^*X^\rho E^fM_{\mathfrak{st}}X^\kappa S_d + \mathscr{W}_{r,n}^{\rhd\lambda}) = (S_d^*X^\kappa E^fM_{\mathfrak{ts}}X^\rho S_e + \mathscr{W}_{r,n}^{\rhd\lambda})a^* \in W,$$

by Proposition 7.13. Hence, W is closed under left multiplication by elements of $\mathscr{W}_{r,n}$. $\square$

Let $\Lambda_r^+ = \{ (f,\lambda) \mid 0 \le f \le \lfloor \frac{n}{2} \rfloor \text{ and } \lambda \in \Lambda_r^+(n-2f) \}$. If $(f,\lambda) \in \Lambda_r^+$ and $(\mathfrak{s},\rho,e),(\mathfrak{t},\kappa,d) \in \delta(f,\lambda)$ then we define

$$C^{(f,\lambda)}_{(\mathfrak{s},\rho,e)(\mathfrak{t},\kappa,d)} = S_e^* X^\rho E^f M_{\mathfrak{s}\mathfrak{t}} X^\kappa S_d.$$

We can now prove Theorem B from the introduction.

Theorem 7.15. *Let R be a commutative ring in which 2 is a unit and let $\mathbf{u} \in R^r$. Suppose that Ω is $\mathbf{u}$–admissible. Then*

$$\mathscr{C} = \{ C^{(f,\lambda)}_{(\mathfrak{s},\rho,e)(\mathfrak{t},\kappa,d)} \mid (\mathfrak{s},\rho,e),(\mathfrak{t},\kappa,d) \in \delta(f,\lambda), \text{ where } (f,\lambda) \in \Lambda_r^+ \}$$

is a cellular basis of $\mathscr{W}_{r,n}(\mathbf{u})$.

Proof. Applying the definitions is easy to check that $(C^{(f,\lambda)}_{(\mathfrak{s},\rho,e)(\mathfrak{t},\kappa,d)})^* = C^{(f,\lambda)}_{(\mathfrak{t},\kappa,d)(\mathfrak{s},\rho,e)}$. Furthermore, by Proposition 7.13, for each $h \in \mathscr{H}$ there exist scalars $r_{(\mathfrak{t}',\kappa',d')}(h) \in R$, which do not depend on $(\mathfrak{s},\rho,e)$, such that

$$C^{(f,\lambda)}_{(\mathfrak{s},\rho,e)(\mathfrak{t},\kappa,d)} \cdot h = \sum_{(\mathfrak{t}',\kappa',d') \in \delta(f,\lambda)} r_{(\mathfrak{t}',\kappa',d')}(h) C^{(f,\lambda)}_{(\mathfrak{s},\rho,e)(\mathfrak{t}',\kappa',d')} \pmod{\mathscr{W}_{r,n}^{\rhd\lambda}}.$$

Therefore, in order to show that $\mathscr{C}$ is a cellular basis of $\mathscr{W}_{r,n}$ it remains to check that $\mathscr{C}$ is a basis of $\mathscr{W}_{r,n}$.

Now, $\mathscr{W}_{r,n} = \mathscr{W}_{r,n}^0 \supset \mathscr{W}_{r,n}^1 \supset \cdots \supset \mathscr{W}_{r,n}^{\lfloor \frac{n}{2} \rfloor}$ is a filtration of $\mathscr{W}_{r,n}$ by two–sided ideals, and the two–sided ideals $\mathscr{W}_{r,n,}^{\rhd\lambda}$ where $\lambda \in \Lambda_r^+(n-2f)$, induce a filtration of $\mathscr{W}_{r,n}^f/\mathscr{W}_{r,n}^{f+1}$. Therefore, $\mathscr{C}$ spans $\mathscr{W}_{r,n}$ by Proposition 7.14 and we are reduced to showing that the elements of $\mathscr{C}$ are linearly independent. Now,

$$\#\mathscr{C} = \sum_{(f,\lambda)\in\Lambda_r^+} \#\delta(f,\lambda)^2 = \sum_{(f,\lambda)\in\Lambda_r^+} \Big(\#\mathscr{T}^{std}(\lambda) \cdot r^f \cdot \frac{n!}{(n-2f)!2^f f!} \Big)^2,$$

since $\#D_f = [\mathfrak{S}_n : \mathfrak{S}_{n-2f} \times B_f]$ and $B_f \cong \mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_f$ has order $2^f f!$. Therefore,

$$\begin{aligned}
\#\mathscr{C} &= \sum_{0\le f\le\lfloor\frac{n}{2}\rfloor} \frac{(n!)^2 \cdot r^{2f}}{\big((n-2f)!2^f f!\big)^2} \sum_{\lambda\in\Lambda_r^+(n-2f)} \#\mathscr{T}^{std}(\lambda)^2 \\
&= \sum_{0\le f\le\lfloor\frac{n}{2}\rfloor} \frac{(n!)^2 \cdot r^{2f}}{\big((n-2f)!2^f f!\big)^2} \cdot r^{n-2f}(n-2f)! \\
&= r^n \sum_{0\le f\le\lfloor\frac{n}{2}\rfloor} \frac{(n!)^2}{(n-2f)!2^{2f}(f!)^2} = r^n(2n-1)!!,
\end{aligned}$$

where the last equality uses the second last line of the proof of Lemma 5.1. By Theorem 5.4 $\mathscr{W}_{r,n}$ is a free R–module of rank $r^n(2n-1)!!$, so this implies that the elements of $\mathscr{C}$ are linearly independent. Hence, $\mathscr{C}$ is a cellular basis of $\mathscr{W}_{r,n}$ as required. $\square$

8. Classification of the irreducible $\mathscr{W}_{2,n}$-modules

This section for fields of characteristic different from two we classify the irreducible $\mathscr{W}_{r,n}(\mathbf{u})$–modules in terms of the irreducible $\mathscr{H}_{r,n}(\mathbf{u})$–modules.

We begin with a useful result of Wenzl's.

Lemma 8.1 (Wenzl [Wen88, Propositions 2.1(a) and 2.2(a)]).

a) *Any monomial* $B_\gamma \in \mathscr{B}_n(\omega_0)$ *is either in* $\mathscr{B}_{n-1}(\omega)$ *or it can be written in the form* $a_1\alpha a_2$*, where* $a_i \in \mathscr{B}_{n-1}(\omega_0)$ *and* $\alpha \in \{E_{n-1}, S_{n-1}\}$.

b) $E_{n-1}\mathscr{B}_n(\omega_0)E_{n-1} = \mathscr{B}_{n-2}(\omega_0)E_{n-1}$.

Lemma 8.2. *Suppose that* $n \geq 2$*. Then* $S_{n-1}\mathscr{B}_{n-1}(\omega_0)E_{n-1} = \mathscr{B}_{n-1}(\omega_0)E_{n-1}$.

Proof. If $a \in \mathscr{B}_{n-2}(\omega_0)$ then $S_{n-1}aE_{n-1} = aS_{n-1}E_{n-1} = aE_{n-1}$. Suppose $a \notin \mathscr{B}_{n-2}(\omega_0)$. By Lemma 8.1, we can write $a = a_1\alpha a_2$ with $a_i \in \mathscr{B}_{n-2}(\omega_0)$ and $\alpha \in \{E_{n-2}, S_{n-2}\}$. If $\alpha = E_{n-2}$, then $S_{n-1}aE_{n-1} = a_1S_{n-1}E_{n-2}E_{n-1}a_2 = a_1S_{n-2}a_2E_{n-1}$. If $\alpha = S_{n-2}$ then $S_{n-1}aE_{n-1} = a_1S_{n-1}S_{n-2}E_{n-1}a_2 = a_1E_{n-2}a_2E_{n-1}$. In all cases we have $S_{n-1}aE_{n-1} \in \mathscr{B}_{n-1}(\omega_0)E_{n-1}$. □

Lemma 8.3. *Suppose that* $n \geq 2$*. Then* $E_{n-1}\mathscr{W}_{r,n}E_{n-1} = \mathscr{W}_{r,n-2}E_{n-1}$.

Proof. Let $a = X^\alpha B_\gamma X^\beta$ be an r-regular monomial in $\mathscr{W}_{r,n}$. To prove the Proposition it is enough to show that $E_{n-1}aE_{n-1} \in \mathscr{W}_{r,n-2}E_{n-1}$. We may assume that $\alpha_i = \beta_i = 0$ if $i \leq n-1$ since $X_iE_{n-1} = E_{n-1}X_i$ if $i < n-1$ and $E_{n-1}(X_{n-1}+X_n) = (X_{n-1} + X_n)E_{n-1} = 0$. Hence, we can write $a = X_n^kB_\gamma X_n^l$, where $0 \leq k, l < r$. Further, if $k = 0 = l$ then $E_{n-1}aE_{n-1} \in \mathscr{B}_{n-2}(\omega_0)E_{n-1} \subseteq \mathscr{W}_{r,n-2}E_{n-2}$, so there is nothing to prove. Hence, we may assume that $k + l > 0$. By applying the involution $*$, if necessary, we see that there is no loss of generality in supposing that $k > 0$. We split the proof into two cases.

Case 1. $B_\gamma \in \mathscr{B}_{n-1}(\omega_0)$:
If $B_\gamma \in \mathscr{B}_{n-2}(\omega_0)$, then

$$E_{n-1}X_n^kB_\gamma X_n^lE_{n-1} = B_\gamma E_{n-1}X_n^{k+l}E_{n-1} = (-1)^{k+l}B_\gamma E_{n-1}X_{n-1}^{k+l}E_{n-1}.$$

However, $E_{n-1}X_{n-1}^{k+l}E_{n-1} = \omega_{n-1}^{(k+1)}E_{n-1}$ by Lemma 4.15, where $\omega_{n-1}^{(k+l)}$ is a central element in $\mathscr{W}_{r,n-2}$. Hence, the Lemma follows in this case.

Suppose then that $B_\gamma \notin \mathscr{B}_{n-2}(\omega_0)$. Then $B_\gamma = B_{\gamma_1}\alpha B_{\gamma_2}$, with $B_{\gamma_i} \in \mathscr{B}_{n-2}(\omega_0)$ and $\alpha \in \{E_{n-2}, S_{n-2}\}$, so $E_{n-1}X_n^kB_\gamma X_n^lE_{n-1} = B_{\gamma_1}E_{n-1}X_n^k\alpha X_n^lE_{n-1}B_{\gamma_2}$.

First consider the case when $\alpha = E_{n-2}$. Then

$$\begin{aligned}
E_{n-1}X_n^kE_{n-2}X_n^lE_{n-1} &= (-1)^{k+l}E_{n-1}X_{n-1}^kE_{n-2}X_{n-1}^lE_{n-1}\\
&= E_{n-1}X_{n-2}^kE_{n-2}X_{n-2}^lE_{n-1}\\
&= X_{n-2}^kE_{n-1}E_{n-2}E_{n-1}X_{n-2}^l\\
&= X_{n-2}^kE_{n-1}X_{n-2}^l = X_{n-2}^{k+l}E_{n-1}.
\end{aligned}$$

Together with the equations above, this implies the Lemma when $\alpha = E_{n-2}$.

Now suppose that $\alpha = S_{n-2}$. Using the relations (2.1),

$$E_{n-1}X_n^kS_{n-2}X_n^lE_{n-1} = E_{n-1}S_{n-2}X_n^{k+l}E_{n-1} = E_{n-1}E_{n-2}S_{n-1}X_n^{k+l}E_{n-1}.$$

By Lemma 2.3, $S_{n-1}X_n^{k+l} = X_{n-1}^{k+l}S_{n-1} + X$, where $X \in \mathscr{W}_{r,n}$ is a linear combination of terms each of which has total degree in X_n and X_{n-1} strictly less than $k+l$. By induction on $k+l$, $E_{n-1}E_{n-2}XE_{n-1} \in \mathscr{W}_{r,n-2}E_{n-1}$. Further,

$$\begin{aligned}
E_{n-1}E_{n-2}X_{n-1}^{k+l}S_{n-1}E_{n-1} &= E_{n-1}E_{n-2}X_{n-1}^{k+l}E_{n-1} = (-1)^{k+l}E_{n-1}E_{n-2}X_{n-2}^{k+l}E_{n-1}\\
&= (-1)^{k+l}E_{n-1}E_{n-2}E_{n-1}X_{n-2}^{k+l} = (-1)^{k+l}X_{n-2}^{k+l}E_{n-1}.
\end{aligned}$$

Consequently, $E_{n-1}X_n^kB_\gamma X_n^lE_{n-1} \in \mathscr{W}_{r,n-2}E_{n-1}$.

Case 2. $B_\gamma \notin \mathscr{B}_{n-1}(\omega_0)$:
Once again by Lemma 8.1 we can write $B_\gamma = B_{\gamma_1}\alpha B_{\gamma_2}$, where $B_{\gamma_i} \in \mathscr{B}_{n-1}(\omega_0)$ and $\alpha \in \{S_{n-1}, E_{n-1}\}$.

If $\alpha = E_{n-1}$ then, using Case 1 twice,

$$\begin{aligned}E_{n-1}X_n^kB_\gamma X_n^lE_{n-1} &= (-1)^{k+l}E_{n-1}X_{n-1}^kB_{\gamma_1}E_{n-1}B_{\gamma_2}X_{n-1}^lE_{n-1}\\ &= E_{n-1}X_n^kB_{\gamma_1}E_{n-1}B_{\gamma_2}X_n^lE_{n-1}\in \mathscr{W}_{r,n-2}E_{n-1}.\end{aligned}$$

So the Lemma follows when $B_\gamma \notin \mathscr{B}_{n-2}(\omega_0)$ and $\alpha = S_{n-1}$.

Finally, suppose that $\alpha = S_{n-1}$. If $l = 0$ then $S_{n-1}B_{\gamma_2}E_{n-1} = hE_{n-1}$, for some $h \in \mathscr{B}_{n-1}(\omega_0)$, by Lemma 8.2 . Consequently, the result follows from Case 1. Hence, we may assume then that $k > 0$ and that $l > 0$.

Next, suppose that $B_{\gamma_2} \in \mathscr{B}_{n-2}(\omega_0)$. Then

$$E_{n-1}X_n^kB_\gamma X_n^lE_{n-1} = E_{n-1}X_n^kB_{\gamma_1}B_{\gamma_2}S_{n-1}X_n^lE_{n-1}.$$

Note that $S_{n-1}X_n^lE_{n-1} = X_{n-1}^lE_{n-1} + XE_{n-1}$ by Lemma 2.3, where $X \in \mathscr{W}_{r,n}$ has total degree less than $k + l$ in X_n and X_{n-1}. By induction on $k + l$, $E_{n-1}X_n^kB_{\gamma_1}B_{\gamma_2}XE_{n-1} = hE_{n-1}$, for some $h \in \mathscr{W}_{2,n-2}$. Moreover, by Case 1, $E_{n-1}X_n^kB_{\gamma_1}B_{\gamma_2}X_n^lE_{n-1} \in \mathscr{W}_{r,n-2}E_{n-1}$. Hence, if $B_{\gamma_2} \in \mathscr{B}_{n-2}(\omega_0)$ the result now follows.

Now suppose that $B_{\gamma_2} \notin \mathscr{B}_{n-2}(\omega_0)$. Then either $B_{\gamma_2} = B_{\gamma_3}E_{n-2}B_{\gamma_4}$, or $B_{\gamma_2} = B_{\gamma_3}S_{n-2}B_{\gamma_4}$, where $B_{\gamma_3}, B_{\gamma_4} \in \mathscr{B}_{n-2}(\omega_0)$. If $B_{\gamma_2} = B_{\gamma_3}E_{n-2}B_{\gamma_4}$ then

$$E_{n-1}X_n^kB_\gamma X_n^lE_{n-1} = E_{n-1}X_n^kB_{\gamma_1}B_{\gamma_3}S_{n-1}E_{n-2}E_{n-1}X_{n-2}^lB_{\gamma_4},$$

and the Lemma follows by induction because $k < k+l$. If $B_{\gamma_2} = B_{\gamma_3}S_{n-2}B_{\gamma_4}$ then

$$E_{n-1}X_n^kB_\gamma X_n^lE_{n-1} = E_{n-1}X_n^kB_{\gamma_1}B_{\gamma_3}S_{n-1}X_n^lS_{n-2}E_{n-1}B_{\gamma_4}.$$

Now, $S_{n-1}X_n^l = X_{n-1}^lS_{n-1} + Y$, where Y is a linear combination of monomials of degree strictly less than l. Since

$$E_{n-1}X_n^kB_{\gamma_1}B_{\gamma_3}X_{n-1}^lS_{n-1}S_{n-2}E_{n-1} = E_{n-1}B_{\gamma_1}B_{\gamma_3}X_{n-1}^lE_{n-2}E_{n-1}X_{n-2}^k,$$

we have that $E_{n-1}X_n^kB_\gamma X_n^lE_{n-1} \in \mathscr{W}_{r,n-2}E_{n-1}$ by induction.

This completes the proof of Case 2. □

By iterating the Lemma we obtain the result that we really want.

Corollary 8.4. *Suppose $f > 0$, $w \in \mathfrak{S}_n$ and that $\kappa, \rho \in \mathbb{N}_r^{(f)}$. Then*

$$E^fX^\rho S_wX^\kappa E^f = hE^f,$$

for some $h \in \mathscr{W}_{r,n-2f}$.

As we now briefly recall, by the general theory of cellular algebras [GL96,Mat99], every irreducible $\mathscr{W}_{r,n}$–module arises in a unique way as the simple head of some cell module. For each $(f,\lambda) \in \Lambda_r^+$ fix $(\mathfrak{s},\rho,e) \in \delta(f,\lambda)$ and let $C_{(\mathfrak{t},\kappa,d)}^{(f,\lambda)} = C_{(\mathfrak{s},\rho,e)(\mathfrak{t},\kappa,d)}^{(f,\lambda)} + \mathscr{W}_{r,n}^{\rhd\lambda}$. By Theorem 7.15 the cell modules of $\mathscr{W}_{r,n}$ are the modules $\Delta(f,\lambda)$ which are the free R–modules with basis $\{\, C_{(\mathfrak{t},\kappa,d)}^{(f,\lambda)} \mid (\mathfrak{t},\kappa,d) \in \delta(f,\lambda)\,\}$. The cell module $\Delta(f,\lambda)$ comes equipped with a natural bilinear form $\phi_{f,\lambda}$ which is determined by the equation

$$C_{(\mathfrak{s},\rho,e)(\mathfrak{t},\kappa,d)}^{(f,\lambda)}C_{(\mathfrak{t}',\kappa',d')(\mathfrak{s},\rho,e)}^{(f,\lambda)} \equiv \phi_{f,\lambda}\big(C_{(\mathfrak{t},\kappa,d)}^{(f,\lambda)}, C_{(\mathfrak{t}',\kappa',d')}^{(f,\lambda)}\big)\cdot C_{(\mathfrak{s},\rho,e)(\mathfrak{s},\rho,e)}^{(f,\lambda)} \pmod{\mathscr{W}_{r,n}^{\rhd\lambda}}.$$

The form $\phi_{f,\lambda}$ is $\mathscr{W}_{r,n}$–invariant in the sense that $\phi_{f,\lambda}(xa,y) = \phi_{f,\lambda}(x,ya^*)$, for $x,y \in \Delta(f,\lambda)$ and $a \in \mathscr{W}_{r,n}$. Consequently,

$$\operatorname{Rad}\Delta(f,\lambda) = \{\, x \in \Delta(f,\lambda) \mid \phi_{f,\lambda}(x,y) = 0 \text{ for all } y \in \Delta(f,\lambda)\,\}$$

is a $\mathscr{W}_{r,n,}$–submodule of $\Delta(f,\lambda)$ and $D(f,\lambda) = \Delta(f,\lambda)/\operatorname{Rad}\Delta(f,\lambda)$ is either zero or absolutely irreducible.

In exactly the same way, for each multipartition $\lambda \in \Lambda_r^+(n-2f)$ the corresponding cell module $S(\lambda)$ for $\mathscr{H}_{r,n-2f}$, the Specht module of section 6, carries a bilinear form ϕ_λ. The quotient module $D(\lambda) = S(\lambda)/\operatorname{Rad} S(\lambda)$ is either zero or an absolutely irreducible $\mathscr{H}_{r,n-2f}$–module.

We can now prove Theorem C.

Theorem 8.5. *Suppose that R is a field with $2 \nmid \operatorname{char} F$, that Ω is $\mathbf{u}$–admissible and that $\omega_0 \neq 0$. Let $(f,\lambda) \in \Lambda_r^+$. Then $D^{(f,\lambda)} \neq 0$ if and only if $D^\lambda \neq 0$.*

Proof. It is enough to prove that $\phi_{f,\lambda} \neq 0$ if and only if $\phi_\lambda \neq 0$.

First, suppose that $\phi_\lambda \neq 0$. Recall that the Specht module $S(\lambda)$ has basis $\{\, m_\mathfrak{t} \mid \mathfrak{t} \in \mathscr{T}^{std}(\lambda)\,\}$. Then $\phi_\lambda(m_\mathfrak{t}, m_\mathfrak{v}) \neq 0$, for some $\mathfrak{t}, \mathfrak{v} \in \mathscr{T}^{std}(\lambda)$; that is, $m_{\mathfrak{s}\mathfrak{t}} m_{\mathfrak{v}\mathfrak{s}} \notin \mathscr{H}^{\rhd\lambda}_{r,n-2f}$. Let $\underline{0}$ to the zero vector in $\mathbb{N}_r^{(f)}$. Then

$$\begin{aligned}
C^{(f,\lambda)}_{(\mathfrak{s},\rho,e)(\mathfrak{t},\underline{0},1)} C^{(f,\lambda)}_{(\mathfrak{v},\underline{0},1)(\mathfrak{s},\rho,e)} &= S_e^* X^\rho E^f M_{\mathfrak{s}\mathfrak{t}} E^f M_{\mathfrak{v}\mathfrak{s}} X^\rho S_e \\
&= S_e^* X^\rho (E^f)^2 M_{\mathfrak{s}\mathfrak{t}} M_{\mathfrak{v}\mathfrak{s}} X^\rho S_e \\
&\equiv \omega_0^f \phi_\lambda(m_\mathfrak{t}, m_\mathfrak{v}) S_e^* X^\rho E^f M_{\mathfrak{s}\mathfrak{s}} X^\rho S_e \\
&\equiv \omega_0^f \phi_\lambda(m_\mathfrak{t}, m_\mathfrak{v}) C^{(f,\lambda)}_{(\mathfrak{s},\rho,e)(\mathfrak{s},\rho,e)} \pmod{\mathscr{W}^{\rhd\lambda}_{r,n}}.
\end{aligned}$$

Hence, $\phi_{f,\lambda}\big(C^{(f,\lambda)}_{(\mathfrak{t},\underline{0},1)}, C^{(f,\lambda)}_{(\mathfrak{v},\underline{0},1)}\big) = \omega_0^f \phi_\lambda(m_\mathfrak{t}, m_\mathfrak{v}) \neq 0$, so that $\phi_{f,\lambda} \neq 0$.

Now suppose that $\phi_{f,\lambda} \neq 0$. Then there exist $(\mathfrak{u},\alpha,u), (\mathfrak{v},\beta',v) \in \delta(f,\lambda)$ such that $\phi_{f,\lambda}\big(C^{(f,\lambda)}_{\mathfrak{u},\alpha,u)}, C^{(f,\lambda)}_{(\mathfrak{v},\beta,v)}\big) \neq 0$. That is,

$$\begin{aligned}
0 \neq{}& C^{(f,\lambda)}_{(\mathfrak{s},\rho,e)(\mathfrak{u},\alpha,u)} \cdot C^{(f,\lambda)}_{(\mathfrak{v},\beta,v)(\mathfrak{s},\rho,e)} \\
&= S_e^* X^\rho E^f M_{\mathfrak{s}\mathfrak{u}} X^\alpha S_u \cdot S_v^* X^\beta E^f M_{\mathfrak{v}\mathfrak{s}} X^\rho S_e \\
&= S_e^* M_{\mathfrak{s}\mathfrak{u}} X^\rho E^f X^\alpha S_u S_v^* X^\beta E^f M_{\mathfrak{v}\mathfrak{s}} X^\rho S_e \\
&= S_e^* X^\rho M_{\mathfrak{s}\mathfrak{u}} h M_{\mathfrak{v}\mathfrak{s}} E^f X^\rho S_e,
\end{aligned}$$

for some $h \in \mathscr{H}_{r,n-2f}$, by Corollary 8.4. Therefore, there is $h' \in \mathscr{H}_{r,n-2f}$ such that $m_{\mathfrak{s}\mathfrak{u}} h' m_{\mathfrak{v}\mathfrak{s}} \neq 0 \pmod{\mathscr{H}^{\rhd\lambda}_{r,n-2f}}$. Consequently, $\phi_\lambda \neq 0$. This completes the proof of the Theorem. □

We remark that the irreducible representations are the Ariki-Koike algebras are indexed by the $\mathbf{u}$–*Kleshchev multipartitions*; see [Ari01, AM00]. In the special case when $u_i = d_i \cdot 1_R$, for $1 \le i \le r$ and where $0 \le d_i < \operatorname{char} R$, Kleshchev [Kle05] has shown that the simple $\mathscr{H}_{r,n}(\mathbf{u})$–modules are also *labelled* by the $\mathbf{u}$–Kleshchev multipartitions of n. Hence, in this case, the simple $\mathscr{W}_{r,n}(\mathbf{u})$–modules are labelled by the set $\{(f,\lambda)\}$, where $0 \le f \le \lfloor \frac{n}{2} \rfloor$ and λ is a $\mathbf{u}$–Kleshchev multipartition of $n-2f$. By modifying the proof of [DM02, Theorem 1.1], or of [AM00, Theorem 1.3], one can show that under the assumptions of Theorem 8.5 the simple $\mathscr{W}_{r,n}(\mathbf{u})$–modules are always labelled by the $\mathbf{u}$–Kleshchev multipartitions. (Note, however, that we are not claiming that $D^{(f,\lambda)} \neq 0$ if and only if λ is Kleshchev.)

Corollary 8.6. *Suppose that R is a field with $2 \nmid \operatorname{char} F$, that Ω is $\mathbf{u}$–admissible and that $\omega_0 \neq 0$. Then $\mathscr{W}_{r,n}(\mathbf{u})$ is a quasi–hereditary algebra if and only if $\operatorname{char} R > n$ and $\mathbf{u}$ is generic for $\mathscr{H}_{r,n}$ (Definition 6.4).*

Proof. By [GL96, (3.10)], a cellular algebra is quasi–hereditary if and only if each cell module has a simple head. Therefore, $\mathscr{W}_{r,n}$ is a quasi–hereditary algebra if and only if $D^{(f,\lambda)} \neq 0$ for all $(f,\lambda) \in \Lambda_r^+$ and $\mathscr{H}_{r,n-2f}$ is cellular if and only if $D^\lambda \neq 0$ for all $\lambda \in \Lambda_r^+(n-2f)$. Hence, by Theorem 8.5, $\mathscr{W}_{r,n}$ is quasi–hereditary if and only if the algebras $\mathscr{H}_{r,n-2f}$ are all quasi–hereditary, for $0 \le f \le \lfloor \frac{n}{2} \rfloor$. However, the degenerate Hecke algebras are symmetric algebras by [Kle05, Cor. 5.7.4], so they are quasi–hereditary precisely when they are semisimple. Hence the result follows from Theorem 6.11. □

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502. Japan.

E-mail address: `ariki@kurims.kyoto-u.ac.jp`

School of Mathematics and Statistics F07, University of Sydney, NSW 2006, Australia

E-mail address: `a.mathas@maths.usyd.edu.au`

Department of Mathematics, East China Normal University, 200062 Shanghai, P.R. China.

E-mail address: `hbrui@math.ecnu.edu.cn`